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An alternating sum of quantum integers

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An alternating sum of quantum integers is quantized.

We quantize the series

$$\sum_{i=1}^{n}(-1)^i i\,.$$

Let

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \qquad x \in \mathbb{Z}\,,\ q \neq 1\,,$$

be the 2nd quantization of x.

***Theorem* 1.** For any $n, r \in \mathbb{Z}_{\geq 0}$,

$$\sum_{i=n}^{n+2r}(-1)^i[i]_q^{\sim} = (-1)^n[n+r]_q^{\sim}\left([r+1]_q^{\sim} - [r]_q^{\sim}\right). \tag{2}$$

Proof. We prove (2) by induction on r. For $r = 0$, (2) becomes

$$(-1)^n[n]_q^{\sim} = (-1)^n[n]_q^{\sim}\,,$$

which is true. The inductive step amounts to:

$$(-1)^n[n+r]_q^{\sim}\left([r+1]_q^{\sim} - [r]_q^{\sim}\right) + (-1)^{n+2r+1}[n+2r+1]_q^{\sim} + (-1)^{n+2r+2}[n+2r+2]_q^{\sim} \stackrel{?}{=} (-1)^n[n+r+1]_q^{\sim}\left([r+2]_q^{\sim} - [r+1]_q^{\sim}\right),$$

or

$$[n+r]_q^{\sim}\left([r+1]_q^{\sim} - [r]_q^{\sim}\right) + \left([n+2r+2]_q^{\sim} - [n+2r+1]_q^{\sim}\right) \stackrel{?}{=} [n+r+1]_q^{\sim}\left([r+2]_q^{\sim} - [r+1]_q^{\sim}\right). \tag{3}$$

As it stands, (3) is valid for $n, r \in \mathbb{R}$, not just for $n, r \in \mathbb{Z}_{\geq 0}$. We now make the crucial observation that (3) is the sum of *two* identities:

$$[n+r]_q^{\sim}[r+1]_q^{\sim} + [n+2r+2]_q^{\sim} = [n+r+1]_q^{\sim}[r+2]_q^{\sim} \tag{4}$$

and

$$[n+r]^{\sim}_q[r]^{\sim}_q + [n+2r+1]^{\sim}_q = [n+r+1]^{\sim}_q[r+1]^{\sim}_q \,, \tag{5}$$

equation (5) following from (4) by $r \to r-1$, $n \to n-1$.

We now prove(5), in the form:

$$[n+r+1]^{\sim}_q[r+1]^{\sim}_q - [n+r]^{\sim}_q[r]^{\sim}_q = [n+r+r+1]^{\sim}_q$$

In this form it is known, for with $x = n+r$, $y = r$, it becomes:

$$[x+1]^{\sim}_q[y+1]^{\sim}_q - [x]^{\sim}_q[y]^{\sim}_q = [x+y+1]^{\sim}_q \,,$$

a known identity and easy to verify in any case $\forall x, y \in \mathbb{R}$. ■

Consider a few evaluations of (2). Notice that for $n = r = 1$, (2) returns,

$$-[1]^{\sim}_q + [2]^{\sim}_q - [3]^{\sim}_q = -[2]^{\sim}_q\Big([2]^{\sim}_q - 1\Big),$$

for $r = 2$,

$$-[1]^{\sim}_q + [2]^{\sim}_q - [3]^{\sim}_q + [4]^{\sim}_q - [5]^{\sim}_q = [3]^{\sim}_q\Big([3]^{\sim}_q - [2]^{\sim}_q\Big),$$

and for $r = 3$,

$$-[1]^{\sim}_q + [2]^{\sim}_q - [3]^{\sim}_q + [4]^{\sim}_q - [5]^{\sim}_q + [6]^{\sim}_q - [7]^{\sim}_q = -[4]^{\sim}_q\Big([4]^{\sim}_q - [3]^{\sim}_q\Big).$$

Likewise, for $n = 0$, $r = 2$, we have

$$-[1]^{\sim}_q + [2]^{\sim}_q - [3]^{\sim}_q + [4]^{\sim}_q = [2]^{\sim}_q\Big([3]^{\sim}_q - [2]^{\sim}_q\Big),$$

and for $r = 3$,

$$-[1]^{\sim}_q + [2]^{\sim}_q - [3]^{\sim}_q + [4]^{\sim}_q - [5]^{\sim}_q + [6]^{\sim}_q = [3]^{\sim}_q\Big([4]^{\sim}_q - [3]^{\sim}_q\Big).$$

Finally, for $n = 5$ and $r = 3$, (2) returns:

$$-[5]^{\sim}_q + [6]^{\sim}_q - [7]^{\sim}_q + [8]^{\sim}_q - [9]^{\sim}_q + [10]^{\sim}_q - [11]^{\sim}_q = -[8]^{\sim}_q\Big([4]^{\sim}_q - [3]^{\sim}_q\Big).$$