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On alternating sums of squares of quantum integers

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The classical formula $\sum_{i=0}^{n}(-1)^{i-1}i^2 = (-1)^{n-1}\sum_{k=0}^{n} i$ is quantized.

The classical formula in the Abstract can be quantized thusly:

$$\sum_{i=0}^{n}(-1)^{i-1}\left([i]_q^{\sim}\right)^2 = (-1)^{n-1}\sum_{i=0}^{n}[i]_{q^2}^{\sim} . \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}} , \tag{2}$$

To prove (1) we set $z_k = q^k - q^{-k}$, notice that (1) is true for $n = 0, 1$, and then proceed by induction. The inductive step $n \to n+2$ amounts to:

$$(-1)^n\left([n+1]_q^{\sim}\right)^2 + (-1)^{n-1}\left([n+2]_q^{\sim}\right)^2 \overset{?}{=} (-1)^{n-1}\left\{[n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim}\right\},$$

or

$$-\left([n+1]_q^{\sim}\right)^2 + \left([n+2]_q^{\sim}\right)^2 \overset{?}{=} [n+1]_{q^2}^{\sim} + [n+2]_{q^2}^{\sim} , \tag{3}$$

Call

$$X = q^n .$$

Then (3) becomes

$$\frac{1}{{z_1}^2}\left\{-\left(Xq - X^{-1}q^{-1}\right)^2 + \left(Xq^2 - X^{-1}q^{-2}\right)^2\right\} \overset{?}{=} \frac{1}{z_2}\left\{X^2q^2 - X^{-2}q^{-2} + X^2q^4 - X^{-2}q^{-4}\right\} \tag{4}$$

The LHS of (4) returns:

$$\frac{1}{{z_1}^2}\left\{(-X^2q^2 + 2 - X^{-2}q^{-2}) + X^2q^4 - 2 + X^{-2}q^{-4}\right\}$$

$$= \frac{1}{{z_1}^2}\left\{X^2q^2(q^2-1) - X^{-2}q^{-2}(1-q^2)\right\}$$

$$= \frac{1}{{z_1}^2}\left\{X^2q^2q(q-q^{-1}) - X^{-2}q^{-2}q^{-1}(q-q^{-1})\right\}$$

$$= \frac{1}{z_1}\left\{X^2q^3 - X^{-2}q^{-3}\right\} = [2n+3]_q^{\sim} . \tag{5}$$

The RHS of (4) yiselds:

$$\frac{1}{z_2}\Big\{X^2q^2(1+q^2) - X^{-2}q^{-2}(1+q^{-2})\Big\}$$

$$= \frac{1}{z_2}\Big\{X^2q^2q(q^{-1}+q) - X^{-2}q^{-2}q^{-1}(q+q^{-1})\Big\}$$

$$= \frac{[2]_q^{\sim}}{z_2}\Big\{X^2q^3 - X^{-2}q^{-3}\Big\} = \frac{1}{z_1}\Big\{X^2q^3 - X^{-2}q^{-3}\Big\} = [2n+3]_q^{\sim}$$

which is the same as (5). We used in the Proof the obvious relation

$$[2]_q^{\sim} = \frac{z_2}{z_1}.$$