*J. Sci. Math. Res.* (2011) Vol. 5, pp. 13-14 Published by Centaurus Press
(Received 2009-06-27)

# Quantum binomials in the second quantization

Boris A. Kupershmidt
The University of Tennessee Space Institute, Tullahoma, TN 37388, USA
E-mail: bkupersh@utsi.edu

The classical formula: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, with $n$ a positive integer, is quantized.

---

We quantize the classical formula

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$

We have:

$$1 + x = \sum_{k=0}^{n} \begin{bmatrix} 1 \\ k \end{bmatrix}_q^{\sim} x^k , \tag{1a}$$

$$(1+qx)(1+q^{-1}x) = 1 + [2]_q^{\sim} x + x^2 , \tag{1b}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}} , \tag{2}$$

so that

$$[2]_q^{\sim} = q + q^{-1}, \tag{3a}$$

$$[3]_q^{\sim} = q^2 + 1 + q^{-2}, \ \ldots \tag{3b}$$

We thus expect for an $m$-product (starting with $(1+q^m x)$) to have:

$$\prod_{i=0}^{m} (1 + q^{m-2i} x) = \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q^{\sim} x^k , \tag{4}$$

where:

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]_q^{\sim}}{[k]_q^{\sim}! [m-k]_q^{\sim}!} , \tag{5}$$

and

$$[s]_q^{\sim}! = [1]_q^{\sim} \ldots [s]_q^{\sim} ; \qquad s \in \mathbb{Z}_{\geq 1} ; \quad [0]_q^{\sim}! = 1 \tag{6}$$

We prove formula (4) by induction on $m$. Assuming it is true for the $(m-2)$ product:

$$\prod_{i=0}^{m-2} (1 + q^{m-2-2i} x) = \sum_{k=0}^{m-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_q^{\sim} x^k , \tag{7}$$

and by factoring the first and last factors out of (4), we may write:

$$\prod_{i=0}^{m}(1+q^{m-2i}x) \;=\; (1+q^m x)(1+q^{-m}x)\prod_{i=0}^{m-2}(1+q^{m-2-2i}x)$$

$$= (1+[2]^{\sim}_{q^m}x+x^2)\sum_{k=0}^{m-1}\begin{bmatrix} m-1 \\ k \end{bmatrix}^{\sim}_q x^k \overset{?}{=} \sum_{k=0}^{m+1}\begin{bmatrix} m+1 \\ k \end{bmatrix}^{\sim}_q x^k,$$

which is equivalent to

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}^{\sim}_q \overset{?}{=} \begin{bmatrix} m-1 \\ k \end{bmatrix}^{\sim}_q + [2]_{q^m}\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}^{\sim}_q + \begin{bmatrix} m-1 \\ k-2 \end{bmatrix}^{\sim}_q, \tag{8}$$

which is immediate, based on the triple of formulae:

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}^{\sim}_q = q^k\begin{bmatrix} m \\ k \end{bmatrix}^{\sim}_q + q^{k-m-1}\begin{bmatrix} m \\ k-1 \end{bmatrix}^{\sim}_q, \tag{9a}$$

$$\begin{bmatrix} m \\ k \end{bmatrix}^{\sim}_q = q^{-k}\begin{bmatrix} m-1 \\ k \end{bmatrix}^{\sim}_q + q^{m-k}\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}^{\sim}_q, \tag{9b}$$

$$\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}^{\sim}_q = q^{1-k}\begin{bmatrix} m-2 \\ k-1 \end{bmatrix}^{\sim}_q + q^{m+2-k}\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}^{\sim}_q, \tag{9c}$$

and the formula

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}^{\sim}_q = q^k\begin{bmatrix} m \\ k \end{bmatrix}^{\sim}_q + q^{k-m-1}\begin{bmatrix} m \\ k-1 \end{bmatrix}^{\sim}_q. \tag{10}$$

Substituting formulae (9) into (10), we recover the desired (8). Above, we used the easily verifiable formula

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}^{\sim}_q = q^a\begin{bmatrix} m \\ k \end{bmatrix}^{\sim}_q + q^b\begin{bmatrix} m \\ k-1 \end{bmatrix}^{\sim}_q, \tag{11}$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} k \\ k-m-1 \end{pmatrix} \text{ or } \begin{pmatrix} -k \\ m+1-k \end{pmatrix},$$

which results from the standard identity

$$[x+y]^{\sim}_q = q^y[x]^{\sim}_q + q^{-x}[y]^{\sim}_q = q^{-y}[x]^{\sim}_q + q^x[y]^{\sim}_q,$$

upon dividing (11) by $[m]^{\sim}_q!/\,[k]^{\sim}_q![m+1-k]^{\sim}_q!$.