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A second weighted sum of quantum factorials

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The classical formula:

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is quantized. We shall prove below that

$$\sum_{k=1}^{n}\left\{q^{\binom{k}{2}+1}\left([k]_q^{\sim}\right)^2 + q^{\frac{k^2-5k+2}{2}}\right\}[k]_q^{\sim}! \;=\; q^{\binom{n}{2}}[n]_q^{\sim}[n+1]_q^{\sim}!\,, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q-q^{-1}}\,, \tag{2}$$

$$[k]_q^{\sim}! = [1]_q^{\sim}...[k]_q^{\sim}\,, \qquad k \in \mathbb{Z}_{\geq 1}\,. \tag{3}$$

Indeed, for $n=1$, (1) returns:

$$q+q^{-1} \;=\; (q+q^{-1})\,,$$

which is true. Here, we used the obvious formula

$$[2]_q^{\sim} \;=\; \frac{q^2-q^{-2}}{q-q^{-1}} \;=\; q+q^{-1}.$$

Now, the inductive step applied to (1) produces

$$\left\{q^{\binom{n+1}{2}+1}\left([n+1]_q^{\sim}\right)^2 + q^{\frac{(n+1)^2-5(n+1)+2}{2}}\right\}[n+1]_q^{\sim}!$$

$$\stackrel{?}{=} \left\{q^{\binom{n+1}{2}}[n+1]_q^{\sim}[n+2]_q^{\sim} - q^{\binom{n}{2}}[n]_q^{\sim}\right\}[n+1]_q^{\sim}!.$$

Dividing this by $[n+1]_q^{\sim}!$, we arrive at

$$q^{\binom{n+1}{2}+1}\left([n+1]_q^{\sim}\right)^2 + q^{\frac{(n^2-3n-2)}{2}} \stackrel{?}{=} q^{\binom{n+1}{2}}[n+1]_q^{\sim}[n+2]_q^{\sim} - q^{\binom{n}{2}}[n]_q^{\sim}.\,, \tag{4}$$

Using the easily verified formula

$$[n+2]^{\sim}_q \;=\; q[n+1]^{\sim}_q \;+\; q^{-n-1}\,, \tag{5}$$

the RHS of (4) becomes:

$$\begin{aligned} & q^{\binom{n+1}{2}}[n+1]^{\sim}_q\left\{q[n+1]^{\sim}_q + q^{-n-1}\right\} - q^{\binom{n}{2}}[n]^{\sim}_q \\ &\qquad = q^{\binom{n+1}{2}+1}\left([n+1]^{\sim}_q\right)^2 + q^{\binom{n+1}{2}-n-1}[n+1]^{\sim}_q - q^{\binom{n}{2}}[n]^{\sim}_q \end{aligned}$$

so that (5) simplifies to:

$$\begin{aligned} q^{\frac{(n^2-3n-2)}{2}} &\overset{?}{=} q^{\binom{n+1}{2}-n-1}[n+1]^{\sim}_q - q^{\binom{n}{2}}[n]^{\sim}_q \\ &= q^{\binom{n}{2}}\left\{q^{\binom{n+1}{2}-\binom{n}{2}-n-1}[n+1]^{\sim}_q - [n]^{\sim}_q\right\} \\ &= q^{\binom{n}{2}}\left\{q^{-1}[n+1]^{\sim}_q - [n]^{\sim}_q\right\}. \end{aligned} \tag{6}$$

But obviously,

$$q^{-1}[n+1]^{\sim}_q \;-\; [n]^{\sim}_q \;=\; q^{-n-1}\,,$$

so the RHS of (6) returns:

$$q^{\binom{n}{2}}q^{-n-1} \;=\; q^{\frac{n(n-1)-2n-2}{2}} \;=\; q^{\frac{n^2-3n-2}{2}}\,,$$

which is the LHS of (6).

Remark 7. Exchange in (1) q by q^{-1}, subtract, and divide by $q-q^{-1}$. We get:

$$\sum_{k=1}^{n}\left\{\left[\binom{k}{2}+1\right]^{\sim}_q([k]^{\sim}_q)^2 + \left[\frac{k^2-5k-2}{2}\right]^{\sim}_q\right\}[k]^{\sim}_q! \;=\; \left[\binom{n}{2}\right]^{\sim}_q [n]^{\sim}_q[n+1]^{\sim}_q!$$

This identity is new even in the classical case $q=1$, where it becomes:

$$\sum_{k=1}^{n}\left\{\left[\binom{k}{2}+1\right]k^2 + \frac{k^2-5k-2}{2}\right\}k! \;=\; \binom{n}{2}n(n+1)!$$