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On weighted sums of quantum factorials

Boris A. Kupershmidt
University of Tennessee Space Institute, 411 B.H. Goethert Parkway, Tullahoma, TN 37388, USA

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is quantized. We shall prove that

$$\sum_{k=1}^{n} q^{k(k+1)/2+1} [k]_q^{\sim} [k]_q^{\sim}! \;=\; q^{n(n+1)/2} [n+1]_q^{\sim}! \;-1\,, \tag{1}$$

where

$$[x]_q^{\sim} \;=\; \frac{q^{x} - q^{-x}}{q - q^{-1}}\,, \tag{2}$$

$$[k]_q^{\sim}! = \prod_{i=1}^{k} [k]_q^{\sim}\,, \qquad k \geq 1\,. \tag{3}$$

For $n = 1$, (1) returns:

$$q^2 \;\overset{?}{=}\; q[2]_q^{\sim} - 1 \;=\; q(q + q^{-1}) - 1 \;=\; q^2\,,$$

which is true. Here, we used the fact that

$$[2]_q^{\sim} \;=\; \frac{q^2 - q^{-2}}{q^1 - q^{-1}} \;=\; q + q^{-1}.$$

We use induction on n for (1). The inductive step amounts to the equality

$$\begin{aligned} q^{(n+1)(n+2)/2+1}[n+1]_q^{\sim}[n+1]_q^{\sim}! &\overset{?}{=} \Big[q^{(n+1)(n+2)/2}[n+2]_q^{\sim}! - 1\Big] - \Big[q^{n(n+1)/2}[n+1]_q^{\sim}! - 1\Big] \\ &= q^{(n+1)(n+2)/2}[n+2]_q^{\sim}[n+1]_q^{\sim}! - q^{n(n+1)/2}[n+1]_q^{\sim}! \end{aligned} \tag{4}$$

Dividing this by $q^{(n+1)(n+2)/2}[n+1]_q^{\sim}!$, we arrive at

$$[n+2]_q^{\sim} \;\overset{?}{=}\; q^1[n+1]_q^{\sim} + q^{-(n+1)}[1]_q^{\sim}\,,$$

which is obviously true, since, in general,

$$[a+b]_q^{\sim} \;=\; q^b[a]_q^{\sim} \;+\; q^{-a}[b]_q^{\sim}\,. \tag{5}$$

Remark 6. Replace q in (1) by q^{-1}, subtract, and divide by $q - q^{-1}$. We get

$$\sum_{k=1}^{n}\left[\frac{k(k+1)}{2}+1\right]_q^{\sim}[k]_q^{\sim}[k]_q^{\sim}! \;=\; \left[\frac{n(n+1)}{2}\right]_q^{\sim}[n+1]_q^{\sim}!$$

This identity is new even in the classical case $q = 1$:

$$\sum_{k=1}^{n}\left[\frac{k(k+1)}{2}+1\right]k \cdot k! \;=\; \frac{n}{2}(n+1)(n+1)!$$