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Resolution of Fractional Relaxation and Diffusion Equations by Adomian's Method

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The Adomian decomposition method (ADM) is generalized to solve fractional relaxation and symmetric space-time fractional diffusion equations. For the second equation, ADM will be combined with Fourier transform due to the irregularity of the initial condition. We obtain the same analytical solutions as the traditional transform technique in terms of power series and in terms of Mellin-Barnes integral.

Keywords: Adomian decomposition method, Fractional differential equations, Anomalous diffusion, Fractional relaxation.

1. Introduction

Fractional differential equations have been investigated by many authors due to their frequent appearance in different research areas and engineering applications. There are various methods to obtain their solutions: Fourier-Laplace transform method, power series method, Green function method. As most of them do not have exact solution, several techniques have been introduced such as purely numerical methods and analytico-numerical methods like the Adomian decomposition method.

The main objective of this study is to implement Adomian's method for solving two kinds of fractional differential equations. We obtain their analytical general solutions in terms of power series and in the form of a Mellin-Barnes integral. The Adomian decomposition method has been widely used to solve a large class of linear and nonlinear ordinary and partial differential equations. The solution is considered as the sum of an infinite series, rapidly converging to an accurate solution. The decomposition method provides an analytical solution without any need for linearization or discretization. The plan of the paper is as follows: In §2, we give some basic definitions and results on the Adomian decomposition method (ADM) and on fractional differential equations. §3 is devoted to the resolution of the fractional equations by the ADM. Finally, concluding remarks are given in §4.

2. Preliminaries

In this section, we present the Adomian decomposition method, the two fractional equations describing respectively relaxation and diffusion processes, and some relative results.

2.1 Adomian Decomposition Method

In the 1980's, George Adomian (1983, 1986, 1989) presented the so-called decomposition method to solve linear and nonlinear integro-differential equations. The technique consists of splitting the given equation into linear and nonlinear parts. Then the solution is decomposed in a series of functions where the nonlinear contribution is obtained in the form of "Adomian's polynomials" from its expansion into power series.

To illustrate the method, consider the following general nonlinear system:

$$\left| \begin{array}{ll} Lu(t) + Ru(t) + Nu(t) & = g(t) \\ u(0) & = u_0 \end{array} \right. \tag{1}$$

where L is the highest order derivative which is assumed to be invertible, R is the remaining linear part, N represents a nonlinear operator, and g is a well-behaved function.

Applying the inverse operator L^{-1} to both side of (1), we have

$$u(t) = f(t) - L^{-1}Ru(t) - L^{-1}Nu(t)\,, \tag{2}$$

where $f(t) = u_0 + L^{-1}g(t)$.

The next step is to introduce the series form of the general solution and the nonlinear operator into (2):

$$u = \sum_{n=0}^{\infty} u_n \qquad \text{and} \qquad Nu = \sum_{n=0}^{\infty} A_n \tag{3}$$

The polynomials (A_n) in $(u_1, \ldots, u_n)$ are the Adomian's polynomials generated by:

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{n=0}^{\infty} u_n\right)\right]_{\lambda=0} \tag{4}$$

Therefore, by identification, we obtain the successive terms of the series solution by the following recurrent relation:

$$\left| \begin{array}{ll} u_0 & = f(t) \\ u_{n+1} & = -L^{-1}Ru_n - L^{-1}A_n \end{array} \right. \tag{5}$$

The decomposition series solution generally converges rapidly in real physical problems (Adomian 1994). Cherruault (1989), Abbaoui and Cherruault (1994, 1995) and Himoun *et al.* (1999) proved rigorously the convergence of this technique. Practically, all terms u_n cannot be determined, so the solution will be approximated by a truncated series

$$\phi_n = \sum_{j=0}^{n-1} u_j \tag{6}$$

2.2 Fractional Equations

Relaxation and diffusion processes arise in various domains of applied sciences like fluids mechanics, viscoelasticity, biology, physics. In complex materials, these processes often display deviations from the standard decay. Some examples of such phenomena are stress relaxation and diffusion of viscoelastic materials, charge-carrier transport in amorphous solids, dielectric relaxation of liquids and solids, diffusion in heterogeneous porous media. In this section, we consider a fractional generalization of relaxation and diffusion equations which model these anomalous processes.

The relaxation differential equation of homogeneous type reads

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\lambda u(t)\,, \tag{7}$$

where λ is a positive constant denoting the inverse of some characteristic time.

One way to generalize this equation is to replace the first time derivative by a Caputo fractional derivative ${}_tD_*^{\beta}$ in order to obtain the fractional relation equation:

$${}_tD_*^{\beta}u(t) = -\lambda u(t), \qquad 0 < \beta \le 1 \tag{8}$$

The Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}_tD_*^{\beta}\, f(t) \;=\; {}_tJ^{m-\alpha}\; {}_tD_*^{\beta} f(t)\,(9)$$

$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau, & m-1<\alpha<m \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \alpha = m \end{cases} \tag{10}$$

where ${}_tJ^{\alpha}$ denotes the Riemann-Liouville time fractional integral of order α, and ${}_tD^{m}$ is the usual time derivative of order m.

Assuming the initial condition u(0+) = 1 and applying the technique of Laplace transforms, the solution reads

$$u(t) = E_{\beta}(-\lambda t^{\beta}) \tag{11}$$

with $E_{\beta}(z) = \sum_{k=0}^{\infty} \dfrac{z^k}{\Gamma(\beta k+1)}$ the Mittag-Leffler function.

By standard diffusion equation we mean the partial differential equation

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t)\,. \tag{12}$$

We adopt the symmetric space-time fractional diffusion equation for the generalization of this equation, namely

$${}_tD_*^{\beta}u(t) \;=\; {}_x\nabla^{\alpha}u(x,t)\,. \tag{13}$$

The symmetric Riesz-Feller fractional derivative ${}_x\nabla^{\alpha}$ of order $\alpha > 0$ is defined from its Fourier transform[1]

$$\mathcal{F}\{{}_x\nabla^{\alpha} f(x);k\} = -\left|k\right|^{\alpha}\,\hat{f}(k)\,. \tag{14}$$

With initial condition $u(x,\,t) = \delta(x)$, we deduce from Mainardi *et al.* (2001) the expression of the associate reduced Green function in terms of Mellin-Barnes integral for $x \ge 0$

[1] Let us recall the Fourier transform of a well-behaved function $f(x)$ defined as

$$\hat{f}(k) = \mathcal{F}\{f(x);k\} = \int_{-\infty}^{+\infty} e^{ikx}\, f(x)\mathrm{d}x, \quad k \in \mathbb{R}\,,$$

and the inverse Fourier transform

$$\mathcal{F}^{-1}\{\hat{f}(s);x\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\, f(k)\mathrm{d}k, \quad x \in \mathbb{R}\,.$$

$$U(x) \;=\; u(x,1) \;=\; \frac{1}{\pi\alpha x}\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma(1-s)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right)}\sin\left(\frac{\pi s}{\alpha}\right)x^s\mathrm{d}s \tag{15}$$

A more general form of the space-time fractional diffusion has been recently treated by a number of authors, such as Saichev and Zaslavsky (1997), Gorenflo *et al.* (2000), Mainardi *et al.* (2001). For more details, the interested reader may consult the references cited therein.

3. Resolution

3.1 Fractional Relaxation

Fractional relaxation is often met in viscoelasticity, rheology, seismology, metallurgy, economic and finance (see Mainardi 1997 and references therein). An example of such process is the dynamics of short term interest rates modeled by (more general) fractional relaxation equation (Jaworska 2008).

In this section, we use the Adomian decomposition method to deal with the homogeneous fractional relaxation equation:

$$\left|\begin{aligned} {}_tD_*^{\beta}u(t) \;&=\; -\,\lambda u(t) \\ u(0^+) \;&=\; 1 \end{aligned}\right. \tag{16}$$

To extend the Adomian's method to fractional case, we set for the linear operator the Caputo fractional derivative $L = {}_tD_*^{\beta}$; and its inverse the Riemann-Liouville fractional integral $L^{-1} = {}_tJ^{\beta}$.

Applying L^{-1} to the system (16) leads to a fractional Volterra integral equation

$$u(t) = 1 - \lambda\, {}_tJ^{\beta}u(t) \tag{17}$$

as

$${}_tJ^{\beta}\;{}_tD_*^{\beta}u(x,t) \;=\; {}_tJ^{\beta}\;{}_tJ^{1-\beta}\;{}_tDu(x,t) \;=\; u(x,t) - u_0(x)\,.$$

Then, we replace $u(t)$ with its series form $u(t) = \sum_{n=0}^{\infty} u_n$, so that (17) becomes

$$\sum_{n=0}^{\infty} u_n(t) \;=\; 1 - \lambda\, {}_tJ^{\beta}\sum_{n=0}^{\infty} u_n(t)\,,$$

which yields, by identification,

$$\left|\begin{aligned} u_0 \;&=\; 1 \\ u_{n+1} \;&=\; -\,\lambda\, {}_tJ^{\beta}u_n. \end{aligned}\right. \tag{18}$$

So

$$u_1 \;=\; -\,\lambda\, {}_tJ^{\beta}u_0 \;=\; -\,\lambda\frac{t^{\beta}}{\Gamma(1+\beta)}$$

$$u_2 = -\lambda \; {}_tJ^\beta u_1 = \lambda^2 \frac{t^{2\beta}}{\Gamma(1+\beta)}$$

$$\vdots$$

$$u_n = -\lambda \; {}_tJ^\beta u_{n-1} = \frac{(-\lambda t^\beta)^n}{\Gamma(1+n\beta)}$$

We obtain the solution as

$$u(x,t) = \sum_{n=0}^{\infty} \frac{(-\lambda t^\beta)^n}{\Gamma\big(1+n\beta\big)} = E_\beta(-\lambda t^\beta)\,. \tag{19}$$

3.2 Symmetric Space-Time Fractional Diffusion

The fractional diffusion equation class has been explicitly introduced in physics by Nigmatullin (1986) for modeling diffusion in special types of porous media which exhibit a fractal geometry. These equations describe anomalous transport in many diverse disciplines, including physics, finance, semiconductor research, biology, and hydrogeology (Hilfer 2003, Meerschaert *et al.* 2002, Metzler and Klafter 2000, Scalas *et al.* 2000).

For the application of the Adomian's method, we consider the following problem of Cauchy of the symmetric space-time fractional diffusion equation:

$$\left| \begin{array}{rcl} {}_tD_*^\beta u(x,t) &=& {}_x\nabla^\alpha u(x,t) \\ u(x,0^+) &=& \delta(x) \end{array} \right. \tag{20}$$

For convenience, we combine the Fourier transform and Adomian's methods to get the solution of above system.

By computing the Fourier transform of system (20), we have:

$$\left| \begin{array}{rcl} {}_tD_*^\beta \hat{u}(k,t) &=& -\left|k\right|^\alpha \hat{u}(k,t) \\ \hat{u}(k,0^+) &=& 1 \end{array} \right. \tag{21}$$

Applying L^{-1}, (21) is equal to

$$\hat{u}(k,t) = 1 - \left|k\right|^\alpha \; {}_tJ^\beta \hat{u}(k,t)\,. \tag{22}$$

Setting $\hat{u}(k,t) = \sum_{n=0}^{\infty} \hat{u}_n(k,t)$, we obtain

$$\left| \begin{array}{rcl} \hat{u}_0 &=& 1 \\ \hat{u}_{n+1} &=& 1 - {}_tJ^\beta \hat{u}_n \end{array} \right. \tag{23}$$

So

$$\hat{u}_n = \frac{(-\left|k\right|^\alpha t^\beta)^n}{\Gamma(1+n\beta)}$$

and

$$\hat{u}(k,t) = E_\beta(-\left|k\right|^\alpha t^\beta)\,. \tag{24}$$

After simple calculations, we establish the scaling property of the fundamental solution $u(x,t)$

$$u(ax,bt) = b^{\frac{-\beta}{\alpha}} u\left(\frac{a}{b^{\beta/\alpha}} x, t\right). \tag{25}$$

Introducing the similarity variable $X = x/(t^{\beta/\alpha})$, we can write

$$u(x,t) \;=\; t^{\frac{-\beta}{\alpha}} u\left(\frac{x}{t^{\beta/\alpha}}, t\right) \;=\; t^{\frac{-\beta}{\alpha}} U\left(\frac{x}{t^{\beta/\alpha}}\right), \tag{26}$$

where $U(x) = u(|x|, 1)$ is the reduced Green function of the Cauchy problem (20).

Evaluating the inverse Fourier transform by the Mellin method (Mainardi *et al.* 2001; Marichev 1983) yields

$$U(x) \;=\; \frac{1}{\pi\alpha x}\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right)\sin\left(\frac{\pi s}{\alpha}\right)} x^s \mathrm{d}s\,, \quad 0<\nu=\Re(s)<1\,, \tag{27}$$

which is equivalent to expression (15) obtained by Mainardi *et al.* (2001).

4. Conclusion

In this work, we extend the Adomian decomposition method to solve fractional relaxation and symmetric space-time fractional diffusion equations with linear fractional operator. To handle the initial condition of the second equation, we have combined the Adomian method with the Fourier transform method and obtained the solution in terms of a Mellin-Barnes integral. The results show that the Adomian method gives the analytical solution. Moreover, the computations are simpler and faster than classical techniques.

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