J. Sci. Math. Res. (2010) Vol. 4, pp. 17-18 Published by Centaurus Press
(Received 2009-06-10)

On the sum of inverse quantum factorials

Boris A. Kupershmidt

University of Tennessee Space Institute, 411 B.H. Goethert Parkway, Tullahoma, TN 37388, USA

The classical formula: the sum, from $k = 1$ to n, of $k/(k+1)! = 1 - 1/(n+1)!$ is quantized.

The classical sum, problem 8.2 of Polya and Kilpatrick (1974),

$$\sum_{k=1}^{n} \frac{1}{(k+1)!} = 1 - \frac{1}{(n+1)!} \tag{1}$$

can be quantized thusly:

$$\sum_{k=1}^{n} \frac{q^{\binom{k}{2}-1}[k]_q^{\sim}}{[k+1]_q^{\sim}!} = 1 - \frac{q^{\binom{n+1}{2}}}{[n+1]_q^{\sim}!}, \tag{2}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[k]_q^{\sim}! = [1]_q^{\sim} \ldots [k]_q^{\sim}, \qquad k \in \mathbb{Z}_{\geq 1}; \quad [0]_q^{\sim}! = 1.$$

For $n = 1$, formula (2) yields:

$$\frac{q^{-1}}{[2]_q^{\sim}} = 1 - \frac{q}{[2]_q^{\sim}},$$

which is true because

$$[2]_q^{\sim} = \frac{q^2 - q^{-2}}{q^1 - q^{-1}} = q + q^{-1},$$

We use induction on n to prove (2). The inductive step amounts to the relation

$$\frac{q^{\binom{n+1}{2}-1}[n+1]_q^{\sim}}{[n+2]_q^{\sim}!} \overset{?}{=} 1 - \frac{q^{\binom{n+2}{2}}}{[n+2]_q^{\sim}!} - \left(1 - \frac{q^{\binom{n+1}{2}}}{[n+1]_q^{\sim}!}\right) = \tag{3}$$

$$= -\frac{q^{\binom{n+2}{2}}}{[n+2]_q^{\sim}!} + \frac{q^{\binom{n+1}{2}}}{[n+1]_q^{\sim}!}$$

or

$$q^{\binom{n+2}{2}-1}[n+1]_q^{\sim} \overset{?}{=} -q^{\binom{n+2}{2}} + q^{\binom{n+1}{2}}[n+2]_q^{\sim}.$$

Dividing by $q^{\binom{n+1}{2}}$, we arrive at

$$[n+2]_q^{\sim} \overset{?}{=} q^{-1}[n+1]_q^{\sim} + q^{n+1},$$

which is obvious, because in general,

$$[a+b]_q^{\sim} = q^b[a]_q^{\sim} + q^{-a}[b]_q^{\sim}, \qquad \forall a, b$$

References

Polya, G. & Kilpatrick, J., *The Stanford Mathematics Problem Book*, Teachers College Press (1974).