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Quantum 12 in double variable base

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Classically, $12 = 2 \cdot 2 \cdot 3 = 4 + 8$. We quantize this trivial observation, nontrivially.

Let

$$[x]^{\sim}_{q} = \frac{q^{x} - q^{-x}}{q - q^{-1}}, \qquad x \in \mathbb{R},\ q \neq 1,$$

be the 2nd quantization of x, so that

$$[2]^{\sim}_{q} = q + q^{-1},$$

$$[3]^{\sim}_{q} = q^{2} + 1 + q^{-2}.$$

***Theorem* 1.** For any $a, n \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} [2]^{\sim}_{q}\,[2]^{\sim}_{q^{a}}\,[3]^{\sim}_{q^{n}} &= [a+2]^{\sim}_{q} - [a-2]^{\sim}_{q} + [2n+2+a]^{\sim}_{q} + [2n+2-a]^{\sim}_{q} \\ &\quad - \Big([2n-2+a]^{\sim}_{q} + [2n-2-a]^{\sim}_{q}\Big). \end{aligned} \tag{2}$$

Proof. Since

$$\begin{aligned} [a+2]^{\sim}_{q} - [a-2]^{\sim}_{q} &= [a+2]^{\sim}_{q} + [2-a]^{\sim}_{q} \\ &= [2]^{\sim}_{q^{a}} + [2]^{\sim}_{q}\,. \end{aligned},$$

and

$$[3]^{\sim}_{q^{n}} - 1 = \Big(q^{2n} + 1 + q^{-2n}\Big) - 1 = [2]^{\sim}_{q^{2n}}\,,$$

(2) becomes:

$$[2]^{\sim}_{q}\,[2]^{\sim}_{q^{a}}\,[2]^{\sim}_{q^{2n}} \stackrel{?}{=} [2n+2+a]^{\sim}_{q} + [2n+2-a]^{\sim}_{q} - \Big([2n-2+a]^{\sim}_{q} + [2n-2-a]^{\sim}_{q}\Big). \tag{3}$$

Now,

$$[2n+2+a]^{\sim}_{q} + [2n+2-a]^{\sim}_{q} = [2]^{\sim}_{q^{a}} + [2n+2]^{\sim}_{q}\,,$$

$$[2n-2+a]^{\sim}_{q} + [2n-2-a]^{\sim}_{q} = [2]^{\sim}_{q^{a}} + [2n-2]^{\sim}_{q}\,.$$

Subtracting, we see that the right side of (3) is:

$$\begin{aligned} [2]^{\sim}_{q^{a}}\Big([2n+2]^{\sim}_{q} - [2n-2]^{\sim}_{q}\Big) &= [2]^{\sim}_{q^{a}}\Big([2n+2]^{\sim}_{q} + [-2n+2]^{\sim}_{q}\Big) \\ &= [2]^{\sim}_{q^{a}}\,[2]^{\sim}_{q^{2n}}\,[2]^{\sim}_{q}\,, \end{aligned}$$

and this is exactly the left side of (3).