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On the sum of quantum factorials

Boris A. Kupershmidt
University of Tennessee Space Institute, 411 B.H. Goethert Parkway, Tullahoma, TN 37388, USA

The classical identity 1·1! + ... + *n*·*n*! = (*n* + 1)! - 1 is quantized.

We shall prove that

$$\sum_{i=1}^{n} q^{1+\binom{i+1}{2}} [i]_q^{\sim} [i]_q^{\sim}! = q^{\binom{n+1}{2}} [n+1]_q^{\sim}! - 1, \tag{1}$$

where

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[k]_q^{\sim}! = [1]_q^{\sim} \ldots [k]_q^{\sim}.$$

For $n = 1$, (1) reduced to

$$q^2 = q(q + q^{-1}) - 1,$$

which is true. We next use induction on n. (1) follows from

$$q^{1+\binom{n+2}{2}} [n+1]_q^{\sim} [n+1]_q^{\sim}! \overset{?}{=} q^{\binom{n+2}{2}} [n+2]_q^{\sim}! - 1 - \left(q^{\binom{n+1}{2}} [n+1]_q^{\sim}! - 1 \right).$$

Dividing both sides by $[n+1]_q^{\sim}!$, we arrive at

$$q^{1+\binom{n+2}{2}} [n+1]_q^{\sim} \overset{?}{=} q^{\binom{n+2}{2}} [n+2]_q^{\sim} - q^{\binom{n+1}{2}}.$$

Dividing this by $q^{\binom{n+2}{2}}$, we get

$$q[n+1]_q^{\sim} + q^{-n-1} \overset{?}{=} [n+2]_q^{\sim}, \tag{2}$$

which is true, since

$$[a+b]_q^{\sim} = q^b [a]_q^{\sim} + q^{-a} [b]_q^{\sim}, \qquad \forall a, b$$

returns (2) for $a = n + 1$, $b = 1$.