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Powers of quantized 2

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We derive a quantum decomposition of the nth power of 2.

In the second quantization

$$x \to [x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},$$

$$[2]_q^\sim = q + q^{-1}.$$

***Theorem* 1.** For $n \in \mathbb{Z}_{\geq 1}$, we have:

$$\left([2]_q^\sim\right)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} [n+1-2k]_q^\sim, \tag{2}$$

where

$$[k]_q^\sim ! = [1]_q^\sim ... [k]_q^\sim, \qquad k \in \mathbb{Z}_{\geq 1}; \quad [0]_q^\sim ! = 1.$$

Proof. We have:

$$\left([2]_q^\sim\right)^n = (q + q^{-1})^n = \sum_{k=0}^{n} \binom{n}{k} q^{n-2k}. \tag{3}$$

Denote

$$\left(\sum_i a_i q^i \right)_+ = \sum_{i \geq 0} a_i q^i.$$

It is easy to check that

$$q^{-i} + q^i = [i+1]_q^\sim - [i-1]_q^\sim, \qquad i \in \mathbb{Z}, \tag{4}$$

$$q^0 = [1]. \tag{4a}$$

Thus (3) returns:

$$\left([2]_q^\sim\right)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \left([k+1]_q^\sim - [k-1]_q^\sim \right), \quad n \text{ is odd.} \tag{5a}$$

For n even, $n = 2m$,

$$\left([2]_q^{\sim}\right)^{2m} \;=\; \sum_{k=0}^{m-1}\binom{2m}{k}\left([k+1]_q^{\sim} - [k-1]_q^{\sim}\right) + \binom{2m}{m}[1]_q^{\sim}\,, \tag{5b}$$

and

$$[1]_q^{\sim} = 1\,.$$

These are *not* formulae of the type (2). The latter comes from the following obvious observation:

***Lemma* 7.**

Let $f(q) = \sum a_i q^i = f(q^{-1}) \leftrightarrow a_i = -a_i$. Then

$$\sum_{i=0}^{n} a_i q^{n-2i} \;=\; \sum_{k=0}^{\lfloor n/2 \rfloor}(a_i - a_{i-1})[n+1-2i]_q^{\sim}\,. \tag{8}$$

Proof. It is enough to remember that

$$[n] = \sum_{i=0}^{n-1} q^{n-1-2i}\,, \qquad\qquad n \in \mathbb{Z}_{\geq 1} \tag{9}$$

and subtract from the sum $\sum_{i=0}^{n} q^{n-2i} a_i$ one $[n+1-2i]_q^{\sim}$ term at a time. We also use $a_{-1} = 0$. ■

Applying the Lemma to the sum (3), we arrive at formula (6).

Notice that for a few small n, formula (2) returns:

$$\left([2]_q^{\sim}\right)^{1} = [2]_q^{\sim}\,, \tag{10a}$$

$$\left([2]_q^{\sim}\right)^{2} = [3]_q^{\sim} + 1\,, \tag{10b}$$

$$\left([2]_q^{\sim}\right)^{3} = [4]_q^{\sim} + 2[2]_q^{\sim}\,, \tag{10c}$$

$$\left([2]_q^{\sim}\right)^{4} = [5]_q^{\sim} + 3[3]_q^{\sim} + 2[1]_q^{\sim}\,, \tag{10d}$$