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How to obtain velocity fields from observed streamline patterns

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A flexible method is presented for making quantitative estimates of the velocity field within bounded regions of recirculating flow. Planar and axi-symmetric cases are considered. An expression for the vorticity field in the core of a vortex ring with a large elliptical cross section is given.

1. Introduction

In recent decades, studies showing streamlines within coherent structures have accumulated in numerous sources in the literature. Quantitatively assessing these archived flow fields is challenging because the associated velocity field is not usually reported, and if it is, it is in the form of graphical velocity vectors, not a single equation for the entire region. However, plots of selected velocity profiles along some semi-axis of a vortex are occasionally reported. An explicit expression for the velocity field throughout such regions would allow integral quantities such as kinetic energy to be easily estimated. In addition to such needs, in the mechanical design of, for example, fuel injection systems in trapped vortex combustors for gas turbines (Straub *et al.* 2005; Beér *et al.* 2004), it would be convenient if a family of vortex shapes could be assessed quickly and parametrically. The present work presents a flexible method of making quantitative estimates of the velocity field in bounded regions of recirculating flow.

The general form of the continuity equation is (Morton 2007a):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}\left(\rho\sqrt{g}v^i\right) = 0\,, \tag{1}$$

where $\sqrt{g}$ is the Jacobian determinant of the transformation from the arbitrary (but inertial) coordinate system, whose ith coordinate is x^i, to a rectangular system. In the present study, the following simplification of (1) will be used:

$$\frac{\partial}{\partial x^i}\left(\rho\sqrt{g}v^i\right) = 0\,. \tag{2}$$

Equation (2) is valid for unsteady or compressible flow, but not both. Transforming to streamline coordinates $\overline{x}^i$ renders (2) separable. Then the nonzero component of the velocity tensor in the streamwise direction, $\overline{v}^2$ say, can be found by integrating (2), to obtain:

$$\overline{v}^2 = \frac{h(\overline{x}^1,\overline{x}^3)}{\rho\sqrt{\overline{g}}}\,, \qquad \overline{v}^1, \overline{v}^3 = 0\,. \tag{3}$$

Here, the constant of integration h is a function of at most $\overline{x}^1$ and $\overline{x}^3$, whose directions are any two independent directions not tangent to the streamline direction, $\overline{x}^2$. Any function h

that does not depend on this streamline coordinate automatically leads, according to (3), to a velocity field that conserves mass throughout.

In order to obtain an estimate of the velocity field using (3), some additional condition must be placed on the flow. The first method is to apply a velocity profile (observed, assumed, or otherwise known) on some semi-axis of the vortex; a second method is to place some condition on the vorticity; and a third method is to make an assumption about the circulation within the vortex. If the flow is steady, it can be fully determined by the continuity equation (2) and such a condition. In such cases, the imposed condition effectively replaces the momentum equation. Examples of methods 1 and 2 will be given.

1.1. Method 1: A Condition on the Velocity Profile

A relationship between the contravariant components of the velocity $\bar{v}^k = \mathrm{d}\bar{x}^k/\mathrm{d}t$ in the streamlined coordinate system and the velocity magnitude v can be obtained from the relation:

$$(\boldsymbol{v}\cdot\boldsymbol{v}) = \bar{v}_k\bar{v}^k = \bar{g}_{jk}\bar{v}^j\bar{v}^k\,, \tag{4}$$

where $\bar{g}_{jk}$ is the metric tensor of the transformation from the streamline coordinate system to a Cartesian coordinate system. Since the only nonzero velocity component in the streamline coordinate system is $\bar{v}^2$, we have

$$(\boldsymbol{v}\cdot\boldsymbol{v}) = \bar{g}_{22}\bar{v}^2\bar{v}^2\,. \tag{5}$$

Taking the square root, and substituting (3) into to the result, gives the velocity magnitude:

$$v = \left\|\boldsymbol{v}\right\| = \frac{\sqrt{\bar{g}_{22}}}{\rho\sqrt{\bar{g}}}\,h(\bar{x}^1,\bar{x}^3)\,. \tag{6}$$

In order to find the function h, some assumption must be made about this velocity magnitude along some semi-axis of the vortex.

1.2. Method 2: A Condition on Vorticity

In general, if a coordinate direction $\bar{x}^3$ is chosen so that it is everywhere orthogonal to the streamwise coordinate $\bar{x}^2$, then $\bar{g}_{23} = 0$ in the metric tensor of the streamline coordinate transformation. If, in addition, partial derivatives with respect to $\bar{x}^3$ vanish, then from the definition of vorticity, given by

$$\bar{\omega}^i = \frac{\varepsilon^{ijk}}{\sqrt{\bar{g}}}\frac{\partial}{\partial\bar{x}^j}\left(\bar{g}_{k2}\bar{v}^2\right), \tag{7}$$

we have $\bar{\omega}^1, \bar{\omega}^2 = 0$, and

$$\bar{\omega}^3 = \frac{1}{\sqrt{\bar{g}}}\left[\left(\bar{g}_{22}\bar{v}^2\right)_{,1} - \left(\bar{g}_{21}\bar{v}^2\right)_{,2}\right]. \tag{8}$$

The Christoffel symbols cancel out in (8); therefore, the covariant and partial derivatives can be considered equivalent. Equation (8) can be used to impose a condition on vorticity.

1.3. Method 3: A Condition on Circulation

An alternative condition that can be placed on (3) is that the circulation values for all closed, nested streamlines in a region be equal (Morton 2004). In other words, one requires that, on a given plane, $\bar{x}^3 =$ const. say, which defines a manifold $\bar{S}_3$, the integral average $\bar{\omega}_A$ of the vorticity tensor, given by

$$\overline{\omega}_A = \frac{\Gamma(\overline{x}^1)}{\displaystyle\int d\overline{S}_3(\overline{x}^1)}, \tag{9}$$

be independent of the streamline $\overline{x}^1$ on which it is computed. Applying (9) on two arbitrary but distinct streamlines on $\overline{S}_3$ provides two independent equations with which to determine the functions h and $\overline{\omega}_A$. The assumption implied by writing the circulation as $\Gamma(\overline{x}^1)$ rather than $\Gamma(\overline{x}^1, x^3)$ requires that any manifold $\overline{S}_3$, made by setting $\overline{x}^3$ to a given value, be identical to any other. It does not rule out the existence of flow in the $\overline{x}^3$ direction.

2. Rotational, Planar Elliptical Vortex (Method 1)

2.1. Coordinate System

The rotational vortex with an elliptical cross section may be more prevalent in real life than the circular one. For example, the two counter-rotating vortices in the time-mean wake bubble trailing prismatic bluff bodies are rarely perfect circles. These may be modeled accurately using the rotational vortex with an elliptical cross section (Morton 2007b). This elliptical vortex should not be confused with the elliptical patch of uniform vorticity found by Kirchhoff (1876; see also Lamb 1932, p. 232), wherein the ellipse itself rotates. What is considered here is the rotational flow within an ellipse that remains stationary.

The first task in constructing the velocity field in such a region is to define a coordinate system wherein contours of one of the contravariant components, $\overline{x}^1$ say, of the position vector of a fluid particle coincide with the vortex streamlines. A large number of flow visualization studies, beginning with those of Prandtl, reveal that the eccentricity of the twin vortices in the near wake region of prismatic bodies in cross flow appears to be constant (see, e.g., Prandtl and Tietjens 1934; Van Dyke 1982). Secondly, the mass flow rates passing through both principal axes (as well as any other axis through the origin) must be equal. However, in order for this integration to be performed along a single coordinate (the $\overline{x}^1$ coordinate), it is convenient to make contours of the $\overline{x}^2$ coordinate pass through the origin. A coordinate system having the desired properties is:

$$\left.\begin{aligned} x^1 &= \overline{x}^1 a\cos(\overline{x}^2) \\ x^2 &= \overline{x}^1 b\sin(\overline{x}^2) \\ x^3 &= \overline{x}^3 \end{aligned}\right\} \tag{10}$$

where x^k are the coordinates of a typical fluid particle relative to a rectangular coordinate system, and a and b are, respectively, the semi-major and semi-minor axes of the elliptical vortex. Therefore, both $\overline{x}^1$ and $\overline{x}^2$ are dimensionless. For the present purposes, this coordinate system has two major advantages over the orthogonal coordinate system of confocal ellipses used by Kirchhoff. First, the eccentricity of the ellipse is constant even for a change in the first coordinate. This ensures that the axis ratio of concentric ellipses remains constant as $\overline{x}^1$ and $\overline{x}^2$ vary independently. Second, integration for the mass flow rate through any semi-axis can be performed along lines of constant $\overline{x}^2$ because these lines all pass through the origin. In short, it is a streamline coordinate system. The coordinate system is shown in Figure 1.

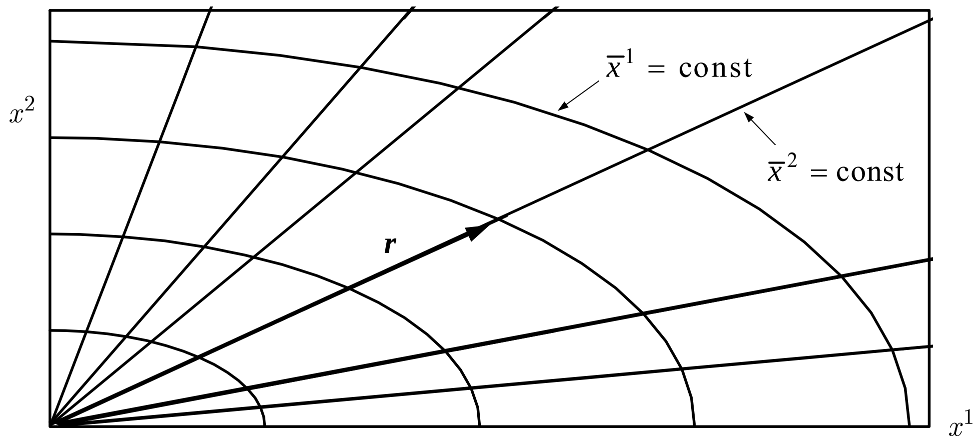

Figure 1. Non-orthogonal, elliptic coordinate system. The curves of constant $\overline{x}^1$ correspond to streamlines of the flow.

The metric tensor for this coordinate system is given by:

$$\overline{g}_{ij} = \sum_k \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^k}{\partial \overline{x}^j}$$

$$\overline{g}_{ij} = \begin{bmatrix} a^2\cos^2(\overline{x}^2) + b^2\sin(\overline{x}^2) & \overline{x}^1(b^2 - a^2)\sin(\overline{x}^2)\cos(\overline{x}^2) & 0 \\ \overline{x}^1(b^2 - a^2)\sin(\overline{x}^2)\cos(\overline{x}^2) & (\overline{x}^1)^2\Big[a^2\sin^2(\overline{x}^2) + b^2\cos^2(\overline{x}^2)\Big] & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

As seen from the off-diagonal elements, the coordinate system is orthogonal only when

$$\overline{x}^2 = 0, \pi/2, \pi, 3\pi/2, ...$$

or when the vortex is circular $(a = b)$. The Jacobian determinant is

$$\sqrt{\overline{g}} = \overline{x}^1 ab. \quad (12)$$

2.2. Velocity Field

With the coordinate system now defined, the magnitude of the physical velocity in (6) can be evaluated along a semi-minor axis, as follows:

$$v\Big|_{(\overline{x}^1,\, \pi/2)} = \left(\frac{\sqrt{\overline{g}_{22}}}{\rho\sqrt{\overline{g}}}\right)\Bigg|_{(\overline{x}^1,\, \pi/2)} h(\overline{x}^1)\,. \quad (13)$$

Evaluating the metric tensor components in (13) using (11) and (12), and solving for the function h gives:

$$h(\overline{x}^1) = \big(\rho v\big)\Big|_{(\overline{x}^1,\, \pi/2)} b\,. \quad (14)$$

The velocity profile in (14) must be zero at the center and reach some maximum on the boundary of the vortex. Experimental measurements of this velocity profile have been made by, among others, Castro and Haque (1987) and Yang and Tsai (1992). Their data show a linear variation in velocity with transverse distance from the vortex center to the perimeter.

If the velocity magnitude along the semi-minor axis is chosen to be a linear function of distance from the origin; that is, if

$$v\Big|_{(\overline{x}^1,\,\pi/2)} = c\,\overline{x}^1\,, \tag{15}$$

then the velocity at the perimeter point is $v_O = c\,\overline{x}^1_{\max}$. Therefore, the constant is:

$$c = \frac{v_O}{\overline{x}^1_{\max}}\,. \tag{16}$$

Substituting (16) into (15) and the result into (14) gives the function h:

$$h(\overline{x}^1) = \rho(\overline{x}^1)\,v_O b\,\frac{\overline{x}^1}{\overline{x}^1_{\max}}\,. \tag{17}$$

Here, the density has been assumed to be a function of only $\overline{x}^1$, so that when (17) is substituted into (3) to find the velocity tensor, density cancels out, leaving:

$$\overline{v}^2 = \frac{v_O}{a}\,. \tag{18}$$

The fluid velocity magnitude can now be calculated from (5).

The velocity distribution may be transformed to the rectangular coordinate system by the transformation

$$v^i = \frac{\partial x^i}{\partial \overline{x}^j}\,\overline{v}^j\,. \tag{19}$$

The result is:

$$\boldsymbol{v} = \begin{bmatrix} v^1 \\ v^2 \\ 0 \end{bmatrix} = \begin{bmatrix} -x^2 \\ x^1\left(\frac{b}{a}\right)^2 \\ 0 \end{bmatrix}\Omega_O\,, \tag{20}$$

where $\Omega_O = v_O/b$. Notice that the maximum velocity occurs at the end of the semi-*minor* axis of the ellipse.

2.3. Vorticity

The vorticity tensor in the streamline coordinate system is computed by (8) to be

$$\overline{\omega}^3 = \Omega_O\left[1 + \left(\frac{b}{a}\right)^2\right]. \tag{21}$$

Therefore, the vorticity within the region is uniform and drops by half in the limit of large axis ratio. Transformation to Cartesian coordinates leaves the value of the uniform vorticity unchanged.

3. Vortex Ring with Plane Elliptical Streamlines (Method 3)

3.1. Coordinate System

A streamline coordinate system for a vortex ring with a large elliptical cross section can be formed as an extension of the elliptical coordinate system used in the previous example. In particular, let (see Morton 2004):

$$x^1 = R\cos\overline{x}^3 \tag{22}$$

$$x^2 = R\sin\overline{x}^3 \tag{23}$$

$$x^3 = \overline{x}^1 a\cos(\overline{x}^2) \tag{24}$$

with

$$R = \overline{x}^1 b\,(\sin(\overline{x}^2) - k) + R_C\,. \tag{25}$$

Here, R_C is the distance from the symmetry axis to the eye of the vortex, k is a constant, and the overbar again represents the streamline coordinate system. Streamlines described by curves of constant $\overline{x}^1$ and $\overline{x}^3$ are concentric when $k = 0$ (see Figure 2). The coordinate transformation represented by (22) through (25) yields a toroidal coordinate system of nested elliptical streamlines with a core circle given by $\overline{x}^1 = 0$. The cross section of the vortex ring is defined by the bounds $(0 \le \overline{x}^1 \le \overline{x}^1_{\max})$ and $(0 \le \overline{x}^2,\ \overline{x}^3 \le 2\pi)$. When $k \ne 0$, the semi-minor axis of the elliptical cross section is not uniquely defined; therefore, the following definition will be imposed: $b \equiv (R_O - R_I)/2$, where R_O and R_I are, respectively, the inner and outer radii of the torus. Consequently, (25) requires that $\overline{x}^1_{\max} = 1$.

The metric tensor is similar to that in the previous example. Its upper diagonal is

$$\overline{g}_{ij} = \begin{bmatrix} a^2\cos^2(\overline{x}^2) + b^2\sin(\overline{x}^2) & \overline{x}^1\Big[(a^2\sin(\overline{x}^2) - b^2(\sin(\overline{x}^2) - k)\Big]\cos(\overline{x}^2) & 0 \\ & (\overline{x}^1)^2\Big[a^2\sin^2(\overline{x}^2) + b^2\cos^2(\overline{x}^2)\Big] & 0 \\ & & R^2 \end{bmatrix}. \tag{26}$$

For this coordinate system, contours of $\overline{x}^3$ are orthogonal to those of $\overline{x}^1$ and $\overline{x}^2$; therefore, $\overline{g}_{13}$ and $\overline{g}_{23}$ are zero, and $\sqrt{\overline{g}_{33}} = R$.

The Jacobian determinant is

$$\sqrt{\overline{g}} = \overline{x}^1 abR(1 - k\sin(\overline{x}^2))\,, \tag{27}$$

and the denominator of (9) is found using the relation $\int \mathrm{d}\overline{S}_3 = \int \sqrt{\overline{g}}\,\mathrm{d}\overline{x}^1\mathrm{d}\overline{x}^2$ to be

$$\overline{S}_3(\overline{x}^1) = \pi ab\overline{x}^1\overline{x}^1\left(R_C - \tfrac{4}{3}bk\,\overline{x}^1\right). \tag{28}$$

3.2. Velocity Field

The assumption that the average given by (9) is independent of the closed streamline within which it is calculated can be stated as follows:

$$\overline{\omega}_A = \frac{\mathit{\Gamma}(\overline{x}^1_{\max})}{\overline{S}_3(\overline{x}^1_{\max})} = \frac{\mathit{\Gamma}(\overline{x}^1)}{\overline{S}_3(\overline{x}^1)}\,, \tag{29}$$

where $\bar{x}^1_{\max}$ designates the outermost streamline (on the perimeter of the vortex) and $\bar{x}^1$ any other streamline. Equation (29) does not require that the vorticity be uniform – only that its average can be calculated solely by geometrical considerations.

The circulation within a closed streamline can be found by applying Stokes' theorem on that streamline, as follows:

$$\mathit{\Gamma}(\bar{x}^1) \;=\; \oint \left(\bar{g}_{22}\bar{v}^2\right)\Big|_{\bar{x}^1} \mathrm{d}\bar{x}^2 \;. \tag{30}$$

Substituting (3) for the velocity field into (30) and noting the dependence of h on only $\bar{x}^1$ and $\bar{x}^3$ gives:

$$\mathit{\Gamma}(\bar{x}^1) \;=\; \frac{h(\bar{x}^1,\bar{x}^3)}{\rho(\bar{x}^1,\bar{x}^3)} \oint \left(\frac{\bar{g}_{22}}{\sqrt{\bar{g}}}\right)\bigg|_{\bar{x}^1,\bar{x}^3} \mathrm{d}\bar{x}^2 \,, \tag{31}$$

provided that either the fluid is incompressible or that the density is constant along each streamline, thereby being independent of $\bar{x}^2$. The assumption made here of a stratified density field will remove any explicit density dependence from the resulting velocity field (see (37)). This assumption that density is stratified by streamline was used implicitly by Moore and Pullin (1998) to obtain numerical and perturbative solutions for a steady compressible spherical vortex. In particular, they used the condition that the quantity $\omega(3)/(R\rho)$ is constant along streamlines, where $\omega(3)$ is the component of the physical vorticity corresponding to the azimuthal direction. For such axisymmetric flows, the quantity $\omega(3)/R$ itself is always constant along streamlines. This is because all components of the vorticity *tensor* not in the streamline direction, such as ω^3, are always conserved along streamlines, provided $\nabla^2\omega = 0$ or the viscosity is zero (see Morton 2007a); and for axisymmetric flows without swirl (such as Hill's spherical vortex), we may write: $\omega(3) = \sqrt{g_{33}}\,\omega^3 = R\omega^3$. Here, g_{33} is a component of the metric tensor of a spherical coordinate system, and ω^3 is the vorticity *tensor* in a spherical coordinate system (see Morton 2004).

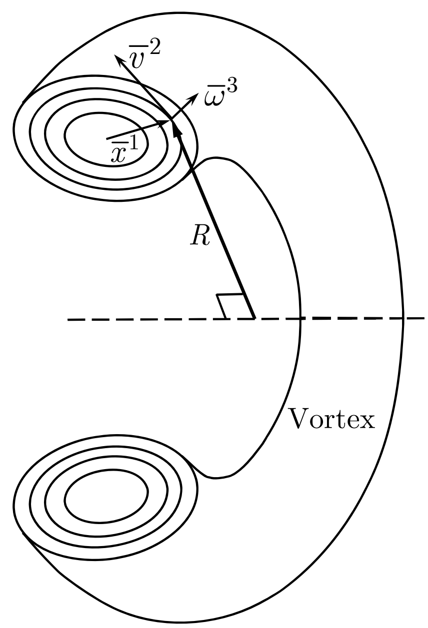

Figure 2. Schematic view of vortex ring with large elliptical cross section.

Returning to the problem at hand, we substitute (31) into (29) twice and solve for $h(\bar{x}^1)$:

$$h(\bar{x}^1) \;=\; h(\bar{x}^1_{\max})\,\frac{\rho(\bar{x}^1)}{\rho(\bar{x}^1_{\max})}\,\frac{\bar{S}_3(\bar{x}^1)}{\bar{S}_3(\bar{x}^1_{\max})}\,\frac{\displaystyle\oint \left(\frac{\bar{g}_{22}}{\sqrt{\bar{g}}}\right)\bigg|_{\bar{x}^1_{\max}} \mathrm{d}\bar{x}^2}{\displaystyle\oint \left(\frac{\bar{g}_{22}}{\sqrt{\bar{g}}}\right)\bigg|_{\bar{x}^1} \mathrm{d}\bar{x}^2} \;. \tag{32}$$

It is interesting to note that without the presence of the velocity in (31), the integrand becomes $\sqrt{\bar{g}_{22}}$, and the integral is the circumference of an ellipse, for which the integral cannot be evaluated directly. (In fact, no exact formula for the circumference of an ellipse has apparently been found, one of the best-known approximations being that of Ramanujan (1914).) However, when the velocity is included, as in (31), the integration in (32) can be performed (see Morton 2004, p. 256). The result is:

$$\oint \left(\frac{\overline{g}_{22}}{\sqrt{\overline{g}}} \right) \bigg|_{\overline{x}^1} \mathrm{d}\overline{x}^2 \;=\; \frac{2\pi \overline{x}^1}{aB^2} F(\overline{x}^1)\,, \tag{33}$$

where

$$F(\overline{x}^1) \;=\; \frac{1}{\overline{x}^1 + kw} \left[\frac{B^2 \overline{x}^1 + w^2/\overline{x}^1}{\sqrt{w^2 - (\overline{x}^1)^2}} \;-\; \frac{w}{\overline{x}^1} \;+\; \frac{B^2 k + 1/k - 1/k\sqrt{1-k^2}}{\sqrt{1-k^2}} \right], \tag{34}$$

$B \equiv b^2/(a^2 - b^2)$, and $w \equiv R_C/b - k\overline{x}^1$. Note that when $k = 0$, the term on the far right is zero.

The value of h on the outermost streamline, required by (32), can be found in terms of the physical velocity v_I at a point on the perimeter by first using (5) to write the physical component of velocity at, say, the point $(\overline{x}^1_{\max}, -\pi/2)$, as follows:

$$v_I \;\equiv\; \left(\sqrt{\overline{g}_{22}}\,\overline{v}^2 \right) \bigg|_{\left(\overline{x}^1_{\max},\, -\frac{\pi}{2}\right)}$$

which, by (3), is

$$v_I \;\equiv\; \left(\sqrt{\overline{g}_{22}}\, \frac{h}{\rho\sqrt{\overline{g}}} \right) \bigg|_{\left(\overline{x}^1_{\max},\, -\frac{\pi}{2}\right)} \;=\; \frac{h(1)}{\rho(1) R_I b (1+k)}\,. \tag{35}$$

Here, the substitution $\overline{x}^1_{\max} = 1$ has been made. Therefore, the value of h on the outermost streamline is:

$$h(1) \;=\; \rho(1) v_I R_I b (1+k)\,. \tag{36}$$

The velocity field can now be determined by substituting (33), (28), and (36) into (32), and the result into (3). Substituting (27) into (3) as well gives the following velocity field in the toroidal coordinate system:

$$\overline{v}^2 = \frac{v_I}{a} \frac{R_I}{R} \frac{F(1)}{F(\overline{x}^1)} \frac{(1-k)\left(R_C - \frac{4}{3} bk\,\overline{x}^1\right)}{\left(1 - k\sin(\overline{x}^2)\right)\left(R_C - \frac{4}{3} bk\right)}\,, \qquad \overline{v}^1,\, \overline{v}^3 = 0 \tag{37}$$

where v_I is the physical velocity $(\sqrt{\overline{g}_{22}}\,\overline{v}^2)$ at the points $R = R_I$ on the torus boundary (farthest from the symmetry axis). Note that the assumption of a stratified density field made in (31) has removed any explicit density dependence from the velocity field in (37), and that the velocity tensor $\overline{v}^2$ is never zero in the torus, even at the eye of the vortex. It is the multiplication by $\sqrt{\overline{g}_{22}}$ to obtain the physical component that reduces the physical velocity to zero at the eye of the vortex. Figure 3 shows a surface plot of velocity magnitude in the core of the vortex ring. Surface contours are projected below it.

When confined to a large elliptical cross section, the resulting flow can provide a snapshot of a centrally-driven vortex ring prior to dissipation of the vorticity (see, e.g., the flow visualization study in Gharib *et al.* 1998). As seen from Figure 3, the peak in velocity occurs at the inside radius of the ring and could therefore model the flow surrounding a strong central jet. It can also be a useful tool in the design of fuel injection systems in gas turbine combustors, wherein a myriad of vortex cross sections must be assessed parametrically.

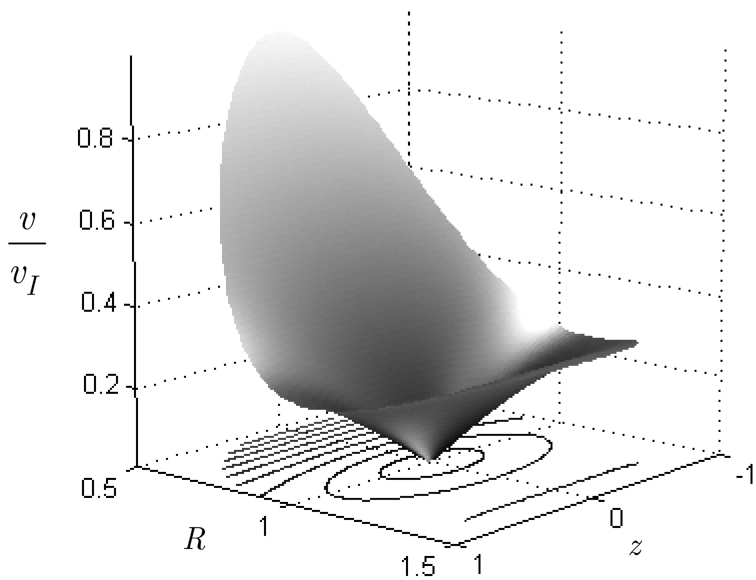

Figure 3. Surface plot of velocity magnitude for the core of a vortex ring with a large elliptical cross section. Contours are projected onto the zero plane. $R_O = 1.5$, $R_C = 1$, $\rho = 1$, $k = 0$.

3.3. Vorticity

The vorticity tensor is computed by (8):

$$\overline{\omega}^3 = \frac{1}{\rho\sqrt{\overline{g}}}\Big[\overline{g}_{22,1}\overline{v}^2 + \overline{g}_{22}\overline{v}^2_{,1} - \overline{g}_{21,2}\overline{v}^2 - \overline{g}_{21}\overline{v}^2_{,2}\Big], \tag{38}$$

where, from (26),

$$\overline{g}_{22,1} = \frac{2\overline{g}_{22}}{\overline{x}^1} \quad \text{and} \quad \overline{g}_{21,2} = \overline{x}^1\Big[(b^2 - a^2)\sin(\overline{x}^2)\cos(\overline{x}^2) - b^2k\cos(\overline{x}^2)\Big].$$

By referring to (37) for the velocity field, the required derivatives of velocity are:

$$\frac{1}{\overline{v}^2}\frac{\partial\overline{v}^2}{\partial\overline{x}^1} = \frac{R_C - \frac{8}{3}bk\,\overline{x}^1}{R_C - \frac{4}{3}bk} - \frac{b\sin(\overline{x}^2)}{R} - \frac{F'}{F}$$

and

$$\frac{1}{\overline{v}^2}\frac{\partial\overline{v}^2}{\partial\overline{x}^2} = -\frac{\overline{x}^1 b\cos(\overline{x}^2)}{R} + \frac{k\cos(\overline{x}^2)}{1 - k\sin(\overline{x}^2)}.$$

Note that when $k = 0$, (34) becomes

$$F(\overline{x}^1) = \frac{B^2 + (w/\overline{x}^1)^2}{\sqrt{w^2 - (\overline{x}^1)^2}} - \frac{w}{(\overline{x}^1)^2}, \qquad (k = 0)$$

so that

$$F' = \frac{2w}{(\overline{x}^1)^3}\left\{1 - \frac{w}{\sqrt{w^2 - (\overline{x}^1)^2}} + \frac{B^2\overline{x}^1 + w^2/\overline{x}^1}{\left[w^2 - (\overline{x}^1)^2\right]^{3/2}}\right\}. \qquad (k = 0)$$

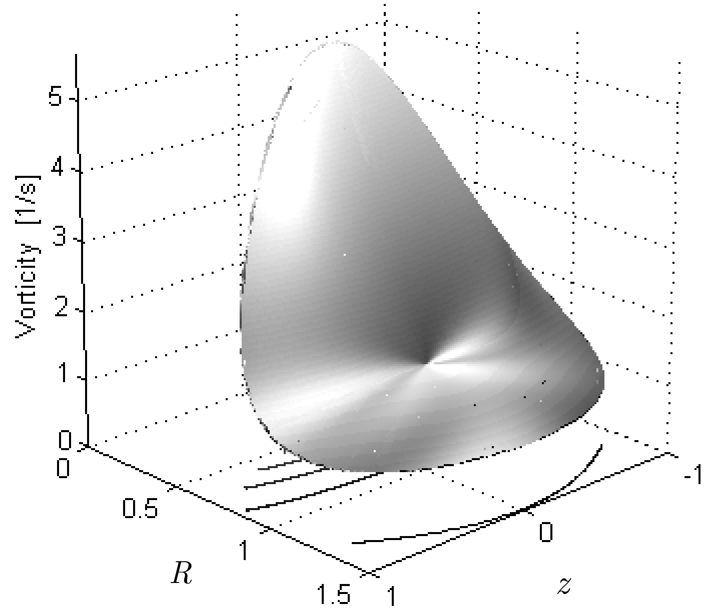

Figure 4. Surface plot of vorticity in the core of a vortex ring with plane elliptical streamlines. $a/b = 2$, $R_O = 1.5$, $R_C = 1$, $k = 0$. Contours of vorticity are shown projected onto the R-z plane.

The vorticity can now be computed by (38) in the form

$$\overline{\omega}^3 = \frac{\overline{v}^2}{\rho\sqrt{\overline{g}}}\left[\overline{g}_{22}\left(\frac{2}{\overline{x}^1} + \frac{1}{\overline{v}^2}\frac{\partial\overline{v}^2}{\partial\overline{x}^1}\right) - \overline{g}_{21,2} - \frac{\overline{g}_{21}}{\overline{v}^2}\frac{\partial\overline{v}^2}{\partial\overline{x}^2}\right]. \qquad (k = 0)$$

The magnitude of vorticity is then given by $\sqrt{\overline{g}_{33}}\overline{\omega}^3 = R\,\overline{\omega}^3$. A surface plot of the vorticity field is given in Figure 4.

4. Conclusion

A fairly flexible method is presented for obtaining the velocity field in bounded regions of recirculating flow. The method is applied to two sample cases: 1) a rotational vortex with an elliptical cross section and 2) a vortex ring with a large elliptical cross section. The examples demonstrate two types of conditions that can be imposed, in lieu of the momentum equation, in order to obtain quick estimates of the velocity field from the continuity equation based on observed streamline patterns. The velocity and vorticity fields were given for a vortex ring having a large elliptical cross section.

References

Beér, J. M., Toqan, M. A., Haynes, J. M., & Borio, R. W., "Development of the Radially Stratified Flame Core Low Nox Burner: From Fundamentals to Industrial Applications," *Journal of Engineering for Gas Turbines and Power* **248** (2004) pp. 248-253.

Castro, I. P. & Haque, A., "The structure of a turbulent shear layer bounding a separation region," *Journal of Fluid Mechanics* **179** (1987) pp. 439-468.

Gharib, M., Rambod, E. & Shariff, K., "A universal time scale for vortex ring formation," *Journal of Fluid Mechanics* **360** (1998) pp. 121-140.

Kirchhoff, G., *Vorlesungen über Matematische Physik* **1**, Leipzig: Teubner (1876) 466 pp.

Lamb, H., *Hydrodynamics*, Dover, NY (1932); (Cambridge University Press, NY, 1993).

Moore, D. W. & Pullin, D. I., "On steady compressible flows with compact vorticity; the compressible Hill's spherical vortex," *Journal of Fluid Mechanics* **374** (1998) pp. 285-303.

Morton, T. S., "The velocity field within a vortex ring with a large elliptical cross-section," *J. Fluid Mech.* **503** (2004) pp. 247-271.

Morton, T. S., "A simplification of the vorticity equation and an extension of the vorticity persistence theorem to three dimensions," *Journal of Scientific and Mathematical Research* **1**, pp. 21-29 (2007a).

Morton, T. S., "A correlation between drag and an integral property of the wake," *Journal of Scientific and Mathematical Research* **1**, pp. 2-20 (2007b).

Prandtl, L. & Tietjens, O. G., *Applied Hydro- and Aeromechanics*, Dover, NY (1934).

Ramanujan, S., "Modular equations and approximations to π," *Quarterly Journal of Mathematics* **45**, 350-372 (1914). See also *Collected Papers*, Cambridge University Press, Cambridge (1927) p. 39; reprinted by Chelsea, New York, (1962); reprinted by the American Mathematical Society, Providence, RI, (2000).

Straub, D. L., Casleton, K. H., Lewis, R. E., Sidwell, T. G., Maloney, D. J., & Richards, G. A., "Assessment of Rich-Burn, Quick-Mix, Lean-Burn Trapped Vortex Combustor for Stationary Gas Turbines," *Journal of Engineering for Gas Turbines and Power* **127** (2005) pp. 36-41.

Van Dyke, M., *An Album of Fluid Motion*, The Parabolic Press, Stanford, CA (1982).

Yang, J.-T., & Tsai, G.-L., "The Wake Flow Structure of an Open-Slit V Gutter," *Experimental Thermal and Fluid Science* **5** (1992) pp. 685-696.