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Quantum binomial coefficients at base 2

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Quantum binomial coefficients are known to be (quantum) integers. In this paper we look at what kind of integers they are in the simplest case.

We prove that, in the second quantization.

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{1}$$

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]_q^{\sim} \ldots [m-k+1]_q^{\sim}}{[1]_q^{\sim} \ldots [k]_q^{\sim}}, \qquad k \in \mathbb{Z}_{\geq 1}, \ m \in \mathbb{Z}, \tag{2}$$

we have:

$$\begin{bmatrix} 2m \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{m-1} [1+4i]_q^{\sim}, \tag{3}$$

where

$$\begin{bmatrix} 2m+1 \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{m-1} [3+4i]_q^{\sim}; \tag{4}$$

these formulae provide a direct proof that $\begin{bmatrix} n \\ 2 \end{bmatrix}_q^{\sim}$ is an integer.

To prove formulae (3) and (4), we appeal to Morton's remarkable formula (Morton 2009)

$$[N]_{q^a}^{\sim} [M]_{q^b}^{\sim} = \sum_{i=0}^{M-1} \Big[N + \frac{b}{a}(1 - M + 2i) \Big]_{q^a}^{\sim}; \tag{5}$$

for $a = 1$, $b = 2$, it simplifies to

$$[N]_q^{\sim} [M]_{q^2}^{\sim} = \sum_{i=0}^{M-1} \big[N + 2(1-M) + 4i \big]_q^{\sim}. \tag{6}$$

We start with formula (3). For the LHS, we get:

$$\begin{bmatrix} 2m \\ 2 \end{bmatrix}_q^{\sim} = \frac{[2m]_q^{\sim} [2m-1]_q^{\sim}}{[2]_q^{\sim}} = [2m-1]_q^{\sim} [m]_{q^2}^{\sim},$$

and by (6),

$$\begin{bmatrix} 2m \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{m-1} \big[m - 1 + 2(1-m) + 4i \big]_q^{\sim}$$

$$= \sum_{i=0}^{m-1} \big[1 + 4i \big]_q^{\sim},$$

which is the RHS of (3).

Similarly, for the LHS of formula (4) we get:

$$\begin{bmatrix} 2m+1 \\ 2 \end{bmatrix}_q^{\sim} = \frac{[2m+1]_q^{\sim}[2m]_q^{\sim}}{[2]_q^{\sim}} = [2m+1]_q^{\sim}[m]_{q^2}^{\sim}$$

and by (6),

$$\begin{bmatrix} 2m+1 \\ 2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{m-1} \big[2m - 1 + 2(1-m) + 4i \big]_q^{\sim}$$

$$= \sum_{i=0}^{m-1} \big[3 + 4i \big]_q^{\sim},$$

which is the RHS of formula (4).

References

Morton, T. S., “A Product of two quantum integers,” *J. Sci. Math. Res.* **3**, pp. 5-6 (2009).