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A binomial form for the sum of consecutive quantum integers

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The classical formula stating that the sum of n consecutive numbers is equal to $(n+1)$ choose n is quantized.

The classical formula $1+2+...+n = \binom{n+1}{2}$ is quantized.
Set

$$[x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{1}$$

The quantized identity is then:

$$\sum_{i=0}^{n} [i]_q^\sim \frac{[2]_{q^i}^\sim}{[2]_q^\sim} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^\sim, \qquad n \in \mathbb{Z}_{>0}, \tag{2}$$

where

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q^\sim = \frac{[n]_q^\sim [n-1]_q^\sim}{[2]_q^\sim}.$$

For $n = 0, 1$, formula (2) is obviously true, because $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1$. We next use induction on n; formula (2) results from

$$\left[n+1\right]_q^\sim \frac{[2]_{q^{n+1}}^\sim}{[2]_q^\sim} \overset{?}{=} \frac{[n+1]_q^\sim}{[2]_q^\sim}\left([n+2]_q^\sim - [n]_q^\sim\right),$$

which is true because

$$[x+2]_q^\sim - [x]_q^\sim = [2]_{q^{x+1}}^\sim. \tag{3}$$

The latter formula is true because, with $z = q - q^{-1}$, the LHS of (3) is:

$$\frac{1}{z}[q^{x+2} - q^{-2-x} - q^x + q^{-x}] =$$

$$= \frac{1}{z}\left[q^x\left(q^2 - 1\right) + q^{-x}\left(1 - q^2\right)\right] = \frac{1}{z}\left[q^x q z + q^{-x} q^{-1} z\right]$$

$$= q^{x+1} + q^{-x-1} = [2]_{q^{x+1}}^\sim,$$

because

$$[2]_q^\sim = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}.$$