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Quantum arithmetic progression whose sums have divisibility properties

Boris A. Kupershmidt and Thad S. Morton
University of Tennessee Space Institute, 411 B.H. Goethert Parkway, Tullahoma, TN 37388, USA

Three new classical formulae involving arithmetic progressions are quantized.

The following classical sum formulae are quantized:

$$\sum_{i=a+1}^{a+rk}(2i) = r\sum_{i=1}^{k}\big(2(ri+a)-r+1\big) \tag{1}$$

$$\sum_{i=a+1}^{a+rk}(2i+1) = r\sum_{i=1}^{k}\big(2(ri+a)-r+2\big) \tag{2}$$

$$\sum_{i=1}^{rk}(2i+b) = r\sum_{i=1}^{k}\big[r(2i-1)+b+1\big] \tag{3}$$

These classical formulae all have the property that their left sides are divisible by r, a rare occurrence. The purpose of this note is to quantize each of these 3 formulae.

Set

$$x \to [x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{4}$$

$$z_k = q^k - q^{-k} \tag{5}$$

***Theorem* 6.** For any $a \in \mathbb{Z}$, $r, k \in \mathbb{Z}_{\geq 1}$, we have:

$$\sum_{i=a+1}^{a+rk}[2i]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}\big[2(ri+a)-r+1\big]_q^{\sim} \tag{7}$$

for the sum of a quantum arithmetic progression.

Proof. We will use the formula (Kupershmidt 2009):

$$[a]_q^{\sim} + [a+d]_q^{\sim} + \ldots + [a+nd]_q^{\sim} = [n+1]_{q^{d/2}}^{\sim}\left[a + \frac{nd}{2}\right]_q^{\sim}. \tag{8}$$

Thus, the left side of (7), having rk terms, and the average value of the term

$$\frac{[2(a+1)] + [2(a+rk)]}{2} = a+1+a+rk = 2a+1+rk, \tag{9}$$

sums to

$$[rk]_q^{\sim}[2a+1+rk]_q^{\sim}\ . \tag{10}$$

For the right side of (7), with k terms, and $d=2r$, the average value of the term is

$$\frac{1}{2}\Big\{[2(a+1)-r+1]+\big[2(rk+a)-r+1\big]\Big\} \;=\; -r+1+r+a+rk+a$$
$$=\; 2a+1+rk,$$

so the right side of (7) is:

$$[r]_q^{\sim}[k]_{q^r}^{\sim}[2a+1+rk]_q^{\sim}$$

which is the same as (10), because, as is well known,

$$[uv]_q^{\sim} = [u]_q^{\sim}[v]_{q^u}^{\sim}\ ,\ \forall u,v\ . \quad \blacksquare \tag{11}$$

***Theorem* 12**. For any $a\in\mathbb{Z}$, $r,k\in\mathbb{Z}_{\geq 1}$, we have:

$$\sum_{i=1+a}^{rk+a}[2i+1]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}\big[2(ri+a)-r+2\big]_q^{\sim}\ . \tag{13}$$

Proof. For the left side, with rk terms, $d=2$, and the average term is

$$\frac{1}{2}\Big\{[2(1+a)+1]+\big[2(rk+a)+1\big]\Big\} \;=\; 1+(1+a)+(rk+a)$$
$$=\; 2a+rk+2.$$

So, the left side is:

$$[rk]_q^{\sim}[2a+rk+2]_q^{\sim}\ , \tag{14}$$

For the right side of (13), $d=2r$, we have k terms, with the average term being

$$\frac{1}{2}\Big\{[2(r+a)-r+2]+\big[2(rk+a)-r+2\big]\Big\} \;=\; \frac{1}{2}\Big\{4a+2rk+4\Big\}$$
$$=\; 2a+rk+2.$$

which is the same as the left side of (14). $\blacksquare$

Remark 15. Formula (7) and (13) are particular cases ($b=0{,}1$) of the following 4-parameter general formula:

$$\sum_{i=1+a}^{rk+a}[2i+b]_q^{\sim} = [r]_q^{\sim}\sum_{i=1}^{k}\big[2(ri+a)-r+b+1\big]_q^{\sim} \tag{16}$$

Indeed, the left side of (16) has rk terms, $d=2$, and the average term is:

$$\frac{1}{2}\Big\{[2(1+a)+b]+\big[2(rk+a)+b\big]\Big\} \;=\; b+(1+a)+(rk+a)$$
$$=\; 2a+rk+1+b,$$

so the left side is

$$[rk]_q^{\sim}[2a + rk + 1 + b]_q^{\sim} \,. \tag{17}$$

The right side has a sum of k terms, $d = 2r$, and the average term is:

$$\frac{1}{2}\Big\{[2(r+a) - r + b + 1] + \Big[2(rk+a) - r + b + 1\Big]\Big\} \;=\; -r + b + 1 + (r+a) + (rk+a)$$
$$=\; 2a + rk + 1 + b.$$

Thus, the right side is

$$[r]_q^{\sim}[k]_{q^r}^{\sim}[2a + rk + 1 + b]_q^{\sim} \,, \tag{18}$$

which is the same as the left side (17).

The third equality (3), results for $a = 0$, $q = 1$ in formula (16).

Notice that by setting $r = 2$ and $a = 0$ in (7), we have, for $k = 1$,

$$[2]_q^{\sim} + [4]_q^{\sim} \;=\; [2]_q^{\sim}[3]_q^{\sim} \,,$$

and for $k = 2$,

$$[2]_q^{\sim} + [4]_q^{\sim} + [6]_q^{\sim} + [8]_q^{\sim} \;=\; [2]_q^{\sim}\Big([3]_q^{\sim} + [7]_q^{\sim}\Big).$$

Likewise, by setting $r = 2$ and $a = 0$ in (13), we have, for $k = 1$,

$$[3]_q^{\sim} + [5]_q^{\sim} \;=\; [2]_q^{\sim}[4]_q^{\sim} \,,$$

and for $k = 2$,

$$[3]_q^{\sim} + [5]_q^{\sim} + [7]_q^{\sim} + [9]_q^{\sim} \;=\; [2]_q^{\sim}\Big([4]_q^{\sim} + [8]_q^{\sim}\Big).$$

By setting $r = 2$, $k = 3$, $b = 1$, and $a = 10$ in (16), we have

$$[23]_q^{\sim} + [25]_q^{\sim} + [27]_q^{\sim} + [29]_q^{\sim} + [31]_q^{\sim} + [33]_q^{\sim} \;=\; [2]_q^{\sim}\Big([24]_q^{\sim} + [28]_q^{\sim} + [32]_q^{\sim}\Big),$$

or, by reversing the values of r and k in (16),

$$[23]_q^{\sim} + [25]_q^{\sim} + [27]_q^{\sim} + [29]_q^{\sim} + [31]_q^{\sim} + [33]_q^{\sim} \;=\; [3]_q^{\sim}\Big([25]_q^{\sim} + [31]_q^{\sim}\Big).$$

References

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