J. Sci. Math. Res. (2009) vol. 3, p. 7 Published by Centaurus Press
(Received 2009-02-20)

On sums of quantum arithmetic progressions

Boris A. Kupershmidt

The University of Tennessee Space Institute, Tullahoma, TN 37388, USA
E-mail: bkupersh@utsi.edu

When the entries of a classical arithmetic progression are replaced by their quantum versions, the resulting sum is still calculable in compact form.

For the classical arithmetic progression, one has:

$$a + (a + d) + (a + 2d) + \dots + (a + nd) = (n + 1)(a + nd/2)\,.$$

Suppose now we replace the numbers $a + kd$ $(k = 0, 1, 2, \dots, n)$ by their quantum versions (in the 2nd quantization):

$$x \to [x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}}\,, \tag{1}$$

getting

$$[a]_q^{\sim} + [a + d]_q^{\sim} + \dots + [a + nd]_q^{\sim}\,. \tag{2}$$

Can the sum (2) be compactly evaluated?

***Theorem* 3.**

$$\sum_{k=0}^{n} [a + kd]_q^{\sim} = [n + 1]_{q^{d/2}}^{\sim} \left[a + n\frac{d}{2}\right]_q^{\sim}. \tag{4}$$

Proof. Multiply both parts of (4) by $(q - q^{-1})$. For the resulting LHS we have:

$$\sum_{k=0}^{n} (q^{a+kd} - q^{-a-kd}) = q^a [n + 1]_{q^d}^{\sim} - q^{-a} [n]_{q^{-d}}\,, \tag{5}$$

where

$$x \to [x]_q = \frac{1 - q^x}{1 - q}\,, \tag{6}$$

is the 1st (standard) quantization. Then (5) turns into

$$q^a \frac{1 - q^{(n+1)d}}{1 - q^d} - q^{-a} \frac{1 - q^{-(n+1)d}}{1 - q^{-d}}$$

$$= q^a \frac{q^{\frac{(n+1)}{2}d}\left(q^{(n+1)d/2} - q^{-(n+1)d/2}\right)}{q^{d/2}\left(q^{d/2} - q^{-d/2}\right)} - q^{-a} \frac{q^{-\frac{(n+1)}{2}d}\left(q^{(n+1)d/2} - q^{-(n+1)d/2}\right)}{q^{-d/2}\left(q^{d/2} - q^{-d/2}\right)}$$

$$= [n + 1]_{q^d}^{\sim} \left\{q^{a+nd/2} - q^{-(a+nd/2)}\right\} = [n + 1]_{q^{d/2}}^{\sim} \left[a + \frac{nd}{2}\right]_{q^d}^{\sim} (q - q^{-1}) \quad \blacksquare$$