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A product of two quantum integers

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The product of two generalized quantum integers is a sum of simple quantum integers.

Quantum integers are generally defined in one of two ways. The first is (see, e.g., Fraenkel 1955):

$$[x]_q = \sum_{i=0}^{x-1} q^i ,$$

so that

$$x \to [x]_q = \frac{1-q^x}{1-q} .$$

In the present work, the second quantization

$$x \to [x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{1}$$

will be used, so that

$$[0]_q^{\sim} = 0 , \quad [1]_q^{\sim} = 1 , \quad [-x]_q^{\sim} = -[x]_q^{\sim} ,$$

and

$$[2]_q^{\sim} = q + q^{-1} , \quad [3]_q^{\sim} = q^2 + 1 + q^{-2} , \ldots$$

$$[n]_q^{\sim} = q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)} , \qquad n \in \mathbb{Z}_{\geq 2} .$$

The quantum integers of the form

$$[n]_q^{\sim} , \qquad n \in \mathbb{Z} \tag{2}$$

are called *simple*, and those of the form

$$[n]_{q^a}^{\sim} , \qquad n \in \mathbb{Z} , \ a \in \mathbb{Z}_{\geq 1} , \tag{3}$$

generalized.

***Theorem* 4.** For any $m, n \in \mathbb{Z}$, $a, b \in \mathbb{Z}_{\geq 1}$,

$$[m]_{q^a}^{\sim} [n]_{q^b}^{\sim} = \sum_{i=0}^{n-1} \Big[m + \frac{b}{a}(1 - n + 2i) \Big]_{q^a}^{\sim} . \tag{5}$$

Proof. Set $\frac{b}{a} = r$, and let $q^a = Q$. Then (5) becomes

$$[m]_Q^{\sim} \ [n]_{Q^r}^{\sim} = \sum_{i=0}^{n-1} \big[m + r(1 - n + 2i) \big]_Q^{\sim} . \tag{6}$$

The left side of (6) is, with $z_k = Q^k - Q^{-k}$, $k \in \mathbb{Z}$, and $N = nr$:

$$\left(\frac{Q^m - Q^{-m}}{Q - Q^{-1}}\right)\left(\frac{Q^{nr} - Q^{-nr}}{z_r}\right) = \frac{1}{z_1 z_r}\left(Q^m - Q^{-m}\right)\left(Q^N - Q^{-N}\right), \tag{7}$$

while, for the right, we get:

$$\sum_{i=0}^{n-1}\left(\frac{Q^{m+r(1-n+2i)} - Q^{-m-r(1-n+2i)}}{Q - Q^{-1}}\right) = \frac{1}{z_1}\left\{Q^{m+r-N}\sum_{i=0}^{n-1} Q^{2ri} - Q^{-m-r+N}\sum_{i=0}^{n-1} Q^{-2ri}\right\}.$$

The sums on the right can be replaced by the first quantization:

$$= \frac{1}{z_1}\left\{Q^{m+r-N}[n]_{Q^{2r}} - Q^{-m-r+N}[n]_{Q^{-2r}}\right\}$$

$$= \frac{1}{z_1}\left\{Q^{m+r-N}\left(\frac{1 - Q^{2nr}}{1 - Q^{2r}}\right) - Q^{-m-r+N}\left(\frac{1 - Q^{-2nr}}{1 - Q^{-2r}}\right)\right\}$$

$$= \frac{1}{z_1}\left\{Q^{m+r-N}\frac{Q^N\left(Q^N - Q^{-N}\right)}{Q^r\left(Q^r - Q^{-r}\right)} - Q^{-m-r+N}\frac{Q^{-N}\left(Q^N - Q^{-N}\right)}{Q^{-r}\left(Q^r - Q^{-r}\right)}\right\}$$

$$= \frac{Q^m - Q^{-m}}{z_1 z_r}\left(Q^N - Q^{-N}\right)$$

which is (7). ■

Note that the right side of formula (5), as it stands, is not a quantum *integer*. This can be remedied as follows: Letting $m = 1$ in (6), we get:

$$[n]^{\sim}_{Q^b} = \sum_{i=0}^{n-1}\left[1 + b(1 - n + 2i)\right]^{\sim}_{Q}\ , \tag{8}$$

and letting $r = 1$ in (6), we get:

$$[m]^{\sim}_{Q}[n]^{\sim}_{Q} = \sum_{i=0}^{n-1}\left[m + 1 - n + 2i\right]^{\sim}_{Q}. \tag{9}$$

Thus, the product of two generalized quantum integers can be expressed by first replacing both factors on the left side of (5) with a series using the right side of (8), giving a sum of products of simple quantum integers. Each product in the sum can then be replaced, using (9), with a sum of simple quantum integers. Therefore, generalized quantum integers form a ring, a result proved previously by Kupershmidt (2009) indirectly.

Remark 10. Formula (5) is true when $m, a, b \in \mathbb{Q}$ or $\mathbb{Z}$; however, these numbers need not be integers.

References

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