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Integrality of the binomial coefficient in the second quantization

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Like in the first (standard) quantization, binomial coefficients turn out to be quantum integers.

1. Introduction

In the 1st (usual) quantization,

$$x \in \mathbb{C} \to [x]_q = \frac{1-q^x}{1-q}, \quad q = e^h,\ h \in \mathbb{C},\ q^x = e^{hx},$$

the binomial coefficients

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]...[m-k+1]}{[1]...[k]}, \quad k \in \mathbb{Z}_{\geq 1},\ m \in \mathbb{Z},$$

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1, \qquad m \in \mathbb{Z},$$

turn out to be, instead of rational functions of q, *polynomials* in q. This follows at once from the formula

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_q = \begin{bmatrix} m \\ k-1 \end{bmatrix}_q + \begin{bmatrix} m \\ k \end{bmatrix}_q q^k \tag{1.1}$$

because

$$[a]_q + q^a [b]_q = [a+b]_q, \quad \forall a, b. \tag{1.1a}$$

The second quantization (symmetric under interchange $q \rightleftarrows q^{-1}$, is:

$$x \to [x]_q^\sim = \frac{q^x - q^{-x}}{q - q^{-1}},$$

so that

$$[-x]_q^\sim = -[x]_q^\sim,$$

$$[n]_q^\sim = q^{n-1} + q^{n-3} + ... + q^{-(n-1)}, \quad n \in \mathbb{Z}_{\geq 2},$$

$$[0]_q^\sim = 0, \quad [1]_q^\sim = 1.$$

The 2nd quantization and the 1st quantization are related, which may explain why the 2nd quantization has been rarely used so far:

$$[x]_q^{\sim} = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{q^{-x}(q^{2x} - 1)}{q^{-1}(q^2 - 1)} = q^{1-x}[x]_{q^2} \tag{1.2}$$

$$\Rightarrow [x]_q = q^{\frac{x-1}{2}}[x]_{q^{1/2}}^{\sim}$$

The *integers* in the 1st quantization, are $\mathbb{Z}$-linear combinations of

$$[n]_{q^i} = 1 + q^i + \ldots + q^{i(n-1)}, \; n \in \mathbb{Z}_{\geq 1}, \; i \in \mathbb{Z},$$

and $[-n]_{q^i} = q^i[n]_{q^i}$. Since

$$-[-1]_{q^{-i}} = q^i, \tag{1.3}$$

the integers in the 1st quantization are of the form

$$\left\{ \sum_{i \in \mathbb{Z}} a_i q^i \mid a_i \in \mathbb{Z}, \text{ finite sums} \right\}$$

These form a *ring*, $\mathcal{O}_1$, because of (1.3).

In the 2nd quantization, the integers are $\mathbb{Z}$-linear combinations of $\{[n]_{q^i}^{\sim}, \; n, i \in \mathbb{Z}\}$ ($i \in \mathbb{Q}$ leads to the isomorphic rings.) The $q \rightleftarrows q^{-1}$ - invariance guarantees that the integers are of the form:

$$\left\{ \sum_{i \in \mathbb{Z}} a_i q^i \,\middle|\, a_{-i} = a_i, \;\; i \in \mathbb{Z}^*; \text{ finite sums.} \right\}.$$

It is immediate that the integers $\mathcal{O}_2$ also form a *ring*, and, in fact,

$$\mathcal{O}_2 = \left\{ \sum a_i q^i \,\middle|\, a_{-i} = a_i, \;\; \forall i; \text{ finite sums} \right\} = \mathcal{O}_1^{\mathrm{Inv}} \subset \mathcal{O}_1$$

The question is: define the binomial coefficients in the 2nd quantization naturally as

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]_q^{\sim} \ldots [m-k+1]_q^{\sim}}{[1]_q^{\sim} \ldots [k]_q^{\sim}}, \qquad m \in \mathbb{Z}, \; k \in \mathbb{Z}_{\geq 0}.$$

Will the integers of $\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim}$ be in $\mathcal{O}_2$? The answer is not obvious, and is in fact, very hard to establish directly. We show that this fact is true in what follows, by going outside $\mathcal{O}_2$, into $\mathcal{O}_1$.

By the way, in the 2nd quantization, an analog of formula (1.1a) is not possible. Instead, we have:

$$[a]_q^{\sim} + [b]_q^{\sim} = [2]_{q^{(b-a)/2}}^{\sim} \left[\frac{a+b}{2} \right]_q^{\sim}$$

so that

$$[z]_q^{\sim} + [z+1]_q^{\sim} = [2z+1]_{q^{1/2}}^{\sim},$$

$$[z+2]_q^{\sim} - [z]_q^{\sim} = [2]_{q^{z+1}}^{\sim},$$

and the symmetry property:

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q^{\sim} = \begin{bmatrix} a+b \\ b \end{bmatrix}_q^{\sim} \qquad a, b \in \mathbb{Z}_{\geq 1}$$

2. The Proof

Denote

$$[k]_q^{\sim}\,! = [k]!_q^{\sim} = [1]_q^{\sim}...[k]_q^{\sim}\,, \quad k \in \mathbb{Z}_{\geq 1}\,, \ \ [0]_q^{\sim}\,! = 1 \tag{2.1}$$

Recall that

$$[k]!_q = [k]_q\,! = [1]_q...[k]_q \qquad k \in \mathbb{Z}_{\geq 1} \tag{2.2}$$

Lemma 2.3. We have:

$$[k]_q^{\sim}! = q^{-\binom{k}{2}}[k]_{q^2}! \qquad k \in \mathbb{Z}_{\geq 1}\,. \tag{2.4}$$

Proof. By formula (1.2),

$$\frac{[k]!_q^{\sim}}{[k]!_{q^2}} = \prod_{i=1}^{k} q^{1-i} = q^k q^{-k(k+1)/2} = q^{-\binom{k}{2}}, \qquad k \in \mathbb{Z}_{\geq 1} \ \ \blacksquare \tag{2.5}$$

Corollary 2.6.

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{k(m+1)}, \qquad k, m \in \mathbb{Z}_{\geq 1}\,.$$

Proof. For $k, m \in \mathbb{Z}_{\geq 1}$, by (2.5),

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = \frac{[m]!_q^{\sim}}{[k]!_q^{\sim}[m-k]!_q^{\sim}} \Rightarrow \frac{\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim}}{\begin{bmatrix} m \\ k \end{bmatrix}_{q^2}} = \frac{q^{-\binom{m}{2}}}{q^{-\binom{k}{2}}}\, q^{-\binom{m-k}{2}}$$

and

$$\binom{k}{2} + \binom{m-k}{2} - \binom{m}{2} = \frac{k^2 - k}{2} + \frac{m^2 - m(2k+1) + k^2 + k}{2} - \frac{m(m-1)}{2}$$

$$= \frac{1}{2}\left(2k^2 - 2mk\right) = -k(m-k)$$

Since $\left[{m \atop k}\right]_{q^2}$ is an integer in $\mathcal{O}_1$, so is $\left[{m \atop k}\right]_q^{\sim}$. But the latter is invariant under $q \rightleftarrows q^{-1}$. So, it belongs to $\mathcal{O}_1^{\mathrm{Inv}} \simeq \mathcal{O}_2$. Thus, $\left[{m \atop k}\right]_q^{\sim}$ is an integer in $\mathcal{O}_2$. This has been proven for positive m; the proof is entirely similar for negative m. Or else, $\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}$.

Remark 2.7. There does not seem to exist in $\mathcal{O}_2$ a relation of the form

$$\begin{bmatrix} m \\ k \end{bmatrix}_q^{\sim} = a\begin{bmatrix} m-1 \\ k \end{bmatrix}_q^{\sim} + b\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q^{\sim}, \qquad a, b \in \mathcal{O}_2$$

which is why a direct proof of the integrality of $\left[{m \atop k}\right]_q^{\sim}$ is so difficult.

Remark 2.8. Curiously,

$$(-1)^k = \begin{bmatrix} -1 \\ k \end{bmatrix}_q^{\sim} = \begin{pmatrix} -1 \\ k \end{pmatrix} \neq \begin{bmatrix} -1 \\ k \end{bmatrix}_q .$$

Remark 2.9. It is easy to see that in $\mathbb{Z}[q, q^{-1}]$,

$$\mathcal{O}_1 = \mathcal{O}_2 \oplus (q - q^{-1})\mathcal{O}_2 .$$